Characterization of Infrared Catastrophe
by The Carleman Operator and Its Singularity

Masao Hirokawa[*]

February 7, 2008

Abstract

This paper addresses some mathematical problems arising from the infrared (IR) catastrophe in quantum field theory. IR catastrophe is formulated and studied in operator theory, characterized by the Carleman operator. Non-existence of ground state under IR catastrophe is also investigated with the help of the characterization. The theory presented in this paper is applied to the Hamiltonian of the model describing a non-relativistic electron coupled with a quantum field of phonons or polaritons in the light of mathematics as well as solid state physics.

1 Introduction

The *infrared (IR) catastrophe* comes up in a wide range of quantum field theory. Each sort of massless quanta makes a quantum field and has a possibility of its causing an individual IR divergence. In the concrete, the divergence of soft photons in quantum electrodynamics, the divergence of soft phonons in solid state physics, the divergence of soft gluons in quantum chromodynamics, etc. In this paper we formulate and handle IR catastrophe with a general framework of operator theory so that we can adapt our method to physical examples as much as possible. Another attempt from this point of view of general aspects was done in [6]. We consider Hamiltonians given by self-adjoint operators acting in a Hilbert space $\mathcal{F}$. Each Hamiltonian H_{QFT} represents the total energy of a physical system coupled with a quantum field. We suppose that H_{QFT} has *IR singularity condition* [4, 5]. The order of the singularity depends on an individual model. So, some of models have IR catastrophe, some not. We express IR catastrophe by the divergence of the ground-state expectation $(\Psi_{\mathrm{QFT}}, N\Psi_{\mathrm{QFT}})_{\mathcal{F}}$ of the total number of bosons, where N is the boson *number operator* acting in $\mathcal{F}$ and Ψ_{QFT} a *ground state* of H_{QFT}. The ground state Ψ_{QFT} is such an eigenvector of H_{QFT} that its eigenvalue is the lowest spectrum of H_{QFT}. The so-called *pull-through formula* [17, 36, 54] is very useful for analyzing IR problems as well as for

studying other problems in quantum field theory (see the literatures in the references of [36]). An idea to obtain the pull-through formula in operator theory was presented in [26] and it was completed in [27]. Hiroshima showed in [33, Theorem 2.9] that we can derive the *Carleman operator* [56] from the operator-theoretical pull-through formula, and then, he characterized a necessary and sufficient condition for the existence of ground state in the domain of $N^{1/2}$ by the Carleman operator in the case where IR catastrophe does not occur even if H_{QFT} has IR singularity condition. Conversely, we investigate IR catastrophe with the *maximal* Carleman operator in this paper.

Let us summarize our path and results here. In Section 2, we prepare some mathematical tools from quantum filed theory and we press ahead with our method through IR problems adopting Dereziński-Gérard's idea [21]. Their idea is explained in Subsection 1.1 below. In Section 3, we restate Hiroshima's [33, Theorem 2.9] and give another proof (Theorem 3.6). We characterize IR catastrophe by simple properties of the domain of the maximal Carleman operator (Theorems 3.7, 3.9, and 3.10). In Section 4, we present a theorem on IR catastrophe (Theorem 4.5) and two theorems on absence of ground state (Theorems 4.2 and 4.7), using the simple domain properties and extending the notion of IR singularity condition (Definition 4.1). Then, we can obtain Dereziński-Gérard's [13, Lemma 2.6] and our [5, Theorem 3.4] as corollaries of one of the theorems (Corollaries 4.3 and 4.4). We also prove that IR singularity condition prohibits H_{QFT} from making the mass gap. Namely, under IR singularity condition there is no spectral gap between the lowest spectrum (i.e., the *ground state energy*) and the infimum of the essential spectrum of H_{QFT} (Theorem 4.6). Without giving a concrete form of H_{QFT}, we assert all these results in a general framework so that our arguments are self-consistent in operator theory. The method presented in this paper enables us to analyze IR catastrophe by investigating the singularity of the maximal Carleman operator.

Let us briefly mention an application of our theory now. There have been many studies for the *full* model in the so-called non-relativistic quantum electrodynamics under IR singularity condition (see [7, 8, 9, 22, 31, 32, 41] and the literatures in their references). For this full model, there is no risk of its meeting IR catastrophe because it has local gauge invariance and thus it brings about the commutation relation, $i[H_{\mathrm{QFT}}, x] = v$, which cancels IR singularity, where x and v are the position and velocity of a non-relativistic electron respectively (see the explanation in [25, p.212 and p.213] about [9] and also Remark 4.1). Thus, Section 5 of this paper addresses IR catastrophe for the Hamiltonian of the models describing a non-relativistic electron coupled with several types of phonons [37] or polaritons [35]. This Hamiltonian is called the Pauli-Fierz (PF) Hamiltonian [46] by authors of [13, 19, 20]. They have made several pieces of painstaking research on its spectral theory, scattering theory, etc. Their PF Hamiltonian has the Fröhlich interaction [16], so it describes, for example, a non-relativistic electron in a polar crystal [15, 38, 39] much better than the electron coupled with photons in quantum electrodynamics. From this point of view, existence of ground state and spec-

tral properties of PF Hamiltonian was investigated in [43]. Because of this physical situation, we call their PF Hamiltonian the *Lee-Low-Pines (LLP) Hamiltonian* [39, Eq.(1)] in Section 5. We apply our results to LLP Hamiltonian and investigate IR catastrophe for it in the light of mathematics as well as solid state physics. We presented in [27] the possible physical mechanism for the situation that IR catastrophe occurs and then no ground state exists in $\mathcal{F}$. Namely, the size of the quasi particle dressed in the cloud of bosons is swelling as $(\Psi_{\mathrm{QFT}}, N\Psi_{\mathrm{QFT}})_{\mathcal{F}}$ increases, and at last it becomes so large that we cannot observe the particle because the uncertainty of particle's position diverges in the ground state. That is when we lose any ground state in $\mathcal{F}$. In [27] we showed this picture for the so-called Nelson Hamiltonian [44] (i.e., the Gross Hamiltonian [23, 24]), directly adopting the idea of the spatial localization of ground state with exponential decay [22]. Based on this picture, as an application of our theory, we give a criterion for IR catastrophe for LLP Hamiltonian (Remark 5.1 and Theorem 5.5). More precisely, let us set the dispersion relation $\omega(k)$ and the interaction function $1^{<\Lambda}(k)\rho(k)$ as $\omega(k) = |k|^{\mu}$ and $\rho(k) = |k|^{-\nu}$ for $\mu \geq 0$ and $\nu \in \mathbb{R}$, respectively, where $k \in \mathbb{R}^d$ is the momentum of bosons, $\Lambda > 0$ a ultraviolet cutoff, and $1^{<\Lambda}(k)$ denotes the characteristic function of $|k| < \Lambda$. Obeying Spohn's result [55], if $\mu + \nu < d/2$, then LLP Hamiltonian has a ground state. On the other hand, IR catastrophe occurs for the (3-dimensional) Nelson Hamiltonian (i.e., $d = 3$, $\mu = 1$, and $\nu = 1/2$) and then it does not have any ground state [13, 27, 42, 45]. Naturally, this result can be extended to LLP Hamiltonian with the condition, $\max\{(\mu/2)+\nu, \mu+\nu-1\} < d/2 < \mu+\nu$ (see Subsection 1.1 and Example 5.1). Thus, we investigate the non-existence of ground state when d, μ, ν are out of the regions. That is, we give a solution to the problem announced in [27, Remark 2]. Once we know that IR catastrophe occurs and thus there is no ground state in $\mathcal{F}$, we have to use non-Fock representation, which has been studied by [2, 13, 17, 29, 45, 51].

1.1　From Two Dereziński-Gérard's Ideas

When we estimate the ground-state expectation $(\Psi_{\mathrm{QFT}}, N\Psi_{\mathrm{QFT}})_{\mathcal{F}}$ of the total number of bosons, it is convenient to use the pull-through formula in the equation:

$$\|N^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \int_{\mathbb{R}^d} \|a(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 dk, \qquad (1.1)$$

where $a(k)$ denotes the kernel of the so-called annihilation operator. The method to establish Eq.(1.1) in operator theory is well known (see, e.g., [27, 33], and also Proposition 2.4). On the other hand, when the integrand $\|a(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2$ in Eq.(1.1) has a singularity at $k = 0$, whether RHS of Eq.(1.1) converges is not certain. So, in such a case, we employ the following expression instead of Eq.(1.1):

$$\|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \int_{|k|>\varepsilon} \|a(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 dk \qquad (1.2)$$

for every $\varepsilon > 0$, where N_ε is the number operator defined as the second quantization of $1^{>\varepsilon}$, the constant function $1(k) = 1$ cut off within the radius of ε from the origin. Thus, by taking $\varepsilon \to 0$ in Eq.(1.2), we can investigate whether IR catastrophe occurs. This is the Dereziński-Gérard's idea [21] which we adopt in our method, though they did not clearly write it in [13]. We establish Eq.(1.2) in operator theory (Lemma 3.3).

   We note another Dereziński-Gérard's idea in [13] concerned with the decomposition of the plane wave. The typical model which represents the case where IR catastrophe occurs under IR singularity condition is the Nelson model. For the Nelson model the pull-through formula has the expression of

$$a(k)\Psi_{\mathrm{QFT}} = -\left(H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}) + \omega(k)\right)^{-1}\left(1^{<\Lambda}(k)\rho(k)e^{-ikx}\right)\Psi_{\mathrm{QFT}},$$

where $E_0(H_{\mathrm{QFT}})$ is the ground state energy of H_{QFT}. We note that this formula should be mathematically established in a certain sense as in [6, 12, 20, 27]. Because the domain of $a(k)$ is so narrow that $a(k)$ is not closable (see e.g., [27, Remark1]) when regarded as an operator, and moreover, $a(k)\Psi_{\mathrm{QFT}}$ may have the singularity at $k = 0$ now. Another Dereziński-Gérard's idea in [13, Lemma 2.2] is the simple decomposition $e^{-ikx} = 1 + (e^{-ikx} - 1)$. Following their idea, $a(k)\Psi_{\mathrm{QFT}}$ can be decomposed into the dipole-approximated term $J_{\mathrm{dip}}(k)\Psi_{\mathrm{QFT}}$ and the error term $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$, i.e., $a(k)\Psi_{\mathrm{QFT}} = J_{\mathrm{dip}}(k)\Psi_{\mathrm{QFT}} + J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$. We know $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$ is IR-safe (i.e., $J_{\mathrm{err}}(\cdot)\Psi_{\mathrm{QFT}} \in L^2(\mathbb{R}^d; \mathcal{F})$) for the Nelson model ($d = 3$, $\mu = 1$, and $\nu = 1/2$) by using $|e^{-ikx} - 1| \leq |k||x|$. Here, of course, showing this square integrability usually requires that $\Psi_{\mathrm{QFT}} \in D(|x|)$ whenever Ψ_{QFT} exists. Obeying this method, to show the error term $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$ is IR-safe for LLP Hamiltonian, the dimension d is usually restricted from below as $\mu + \nu - 1 < d/2$. In general, it is difficult to show that $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$ is IR-safe for LLP Hamiltonian without this restriction. Under the restriction, whether IR catastrophe occurs (i.e., whether RHS of Eq.(1.1) diverges) depends on whether the dipole-approximated term $J_{\mathrm{dip}}(k)\Psi_{\mathrm{QFT}}$ is IR-divergent (i.e., $J_{\mathrm{dip}}(\cdot)\Psi_{\mathrm{QFT}} \notin L^2(\mathbb{R}^d; \mathcal{F})$). Indeed the error term $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$ becomes IR-safe under the restriction, but the following question arises. How can we prove IR catastrophe and non-existence of ground state when we do not know whether the error term $J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}$ is IR-safe? Namely, how can we remove the restriction on d from below? This question was stated in [27, Remark 2]. This paper addresses this question.

2 Set-ups in Mathematics

In this section we prepare some tools from mathematics for quantum field theory and give our Hamiltonian H_{QFT}. Once we obtain the maximal Carleman operator and its domain properties in Section 3, the almost only thing we do is to analyze the Carleman operator and its singularity.

2.1   Preliminaries

Let $X = (X, \mathcal{A}, \mu)$ be a σ-finite measurable space. Let us denote by X^n n-fold Cartesian product of X. The measure for X^n is naturally given by

$d\mu^n(k_1, \cdots, k_n) := d\mu(k_1) \otimes \cdots \otimes d\mu(k_n)$. Thus, we define the boson Fock space $\mathcal{F}_{\mathrm{b}}(L^2(X))$ over $L^2(X) := L^2(X, \mathcal{A}, \mu)$ by

$$\mathcal{F}_{\mathrm{b}}(L^2(X)) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n L^2(X).$$

Here, $\otimes_{\mathrm{s}}^n L^2(X)$ is the n-fold symmetric tensor product of $L^2(X)$ for $n \in \mathbb{N}$ with convention $\otimes_{\mathrm{s}}^0 L^2(X) := \mathbb{C}$. For $\psi \in \mathcal{F}_{\mathrm{b}}(L^2(X))$, we use the following notation:

$$\psi = \oplus \sum_{n=0}^{\infty} \psi^{(n)}, \quad \psi^{(n)} \in \otimes_{\mathrm{s}}^n L^2(X) \; ; \; n \in \{0\} \cup \mathbb{N}.$$

We often abbreviate $\mathcal{F}_{\mathrm{b}}(L^2(X))$ to $\mathcal{F}_X$ for simplicity in this paper, i.e.,

$$\mathcal{F}_X := \mathcal{F}_{\mathrm{b}}(L^2(X)).$$

We employ the standard norm $\| \quad \|_{\mathcal{F}_X}$ in $\mathcal{F}_X$. We denote by $\| \quad \|_{\mathcal{V}}$ the norm of a Hilbert space $\mathcal{V}$, induced its inner product, throughout this paper.

For each $n \in \{0\} \cup \mathbb{N}$ and every $f \in L^2(X)$, we define an operator $a_X(f) : \otimes_{\mathrm{s}}^n L^2(X) \ni \psi^{(n+1)} \mapsto (a_X(f)\psi)^{(n)} \in \otimes_{\mathrm{s}}^n L^2(X)$ by

$$(a_X(f)\psi)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1} \int_X \overline{f(k)} \psi^{(n+1)}(k, k_1, \cdots, k_n) d\mu(k).$$

We can extend $a_X(f)$ to a closed operator acting in $\mathcal{F}_X$ as

$$a_X(f)\psi := \oplus \sum_{n=0}^{\infty} (a_X(f)\psi)^{(n)},$$

$$D(a_X(f)) := \left\{ \psi \in \mathcal{F}_X \;\middle|\; \sum_{n=0}^{\infty} \| (a_X(f)\psi)^{(n)} \|_{\otimes_{\mathrm{s}}^n L^2(X)}^2 < \infty \right\}.$$

We call $a_X(f)$ the *annihilation operator*. Since we can regard it as an operator-valued distribution, symbolically we often write it as

$$a_X(f) = \int_X \overline{f(k)} a_X(k) d\mu(k)$$

with a kernel $a_X(k)$ of the annihilation operator. We define the *creation operator* $a_X^{\dagger}(f)$ for every $f \in L^2(X)$ by $a_X(f)^*$, the adjoint operator of $a_X(f)$, i.e., $a_X^{\dagger}(f) := a_X(f)^*$. The kernel of $a_X^{\dagger}(f)$ is denoted as $a_X^{\dagger}(k)$ frequently.

Let T be every closable operator densely defined in $L^2(X)$. For $n \in \{0\} \cup \mathbb{N}$ we set $T^{(0)}$ as $T^{(0)} := 0$ and define $T^{(n)} : \otimes_{\mathrm{s}}^n L^2(X) \to \otimes_{\mathrm{s}}^n L^2(X)$ by

$$T^{(n)} := \overline{\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \underset{\substack{\frown \\ j\text{-th}}}{T} \otimes I \otimes \cdots \otimes I}.$$

We denote by $\overline{S}$ the closure of a closable operator S. Then, we define an operator $d\Gamma_X(T)$ acting in $\mathcal{F}_X$ by

$$d\Gamma_X(T) := \bigoplus_{n=0}^{\infty} T^{(n)}.$$

We call $d\Gamma_X(T)$ *second quantization of* T. For the second quantization the following facts are well known:

Proposition 2.1

(i) *If* $T \neq 0$*, then* $d\Gamma_X(T)$ *is unbounded.*

(ii) *If* T *is self-adjoint, then* $d\Gamma_X(T)$ *is also self-adjoint.*

(iii) *Let* T *be non-negative, injective, and self-adjoint. Then, for every* $f \in D(T^{-1/2})$

$$D(d\Gamma_X(T)^{1/2}) \subset D(a_X(f)) \cap D(a_X^\dagger(f)).$$

In addition, for every $\psi \in D(d\Gamma_X(T)^{1/2})$

$$
\begin{aligned}
\|a_X(f)\psi\|_{\mathcal{F}_X} &\leq \|T^{-1/2}f\|_{L^2(X)}\|d\Gamma_X(T)^{1/2}\psi\|_{\mathcal{F}_X}, \\
\|a_X^\dagger(f)\psi\|_{\mathcal{F}_X} &\leq \|T^{-1/2}f\|_{L^2(X)}\|d\Gamma_X(T)^{1/2}\psi\|_{\mathcal{F}_X} \\
&\quad + \|f\|_{L^2(X)}\|\psi\|_{\mathcal{F}_X}.
\end{aligned}
$$

Let 1 stand for the multiplication operator of the constant function $1(k) \equiv 1$ of $k \in X$ now. Then, we define an operator N_X acting in $\mathcal{F}_X$ by

$$N_X := d\Gamma_X(1).$$

Let X be able to be decomposed into the disjoint union of X_1 and X_2, i.e., $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Then, $L^2(X)$ is also decomposed into the direct sum of $L^2(X_1)$ and $L^2(X_2)$, i.e., $L^2(X) = L^2(X_1) \bigoplus L^2(X_2)$. The following proposition is known:

Proposition 2.2 *There is a unique unitary operator* $U : \mathcal{F}_X \equiv \mathcal{F}_b(L^2(X)) \to \mathcal{F}_{X_1} \otimes \mathcal{F}_{X_2} \equiv \mathcal{F}_b(L^2(X_1)) \otimes \mathcal{F}_b(L^2(X_2))$ *such that*

(i) *For the individual Fock vacuum,* $\Omega_X \in \mathcal{F}_X$*,* $\Omega_{X_1} \in \mathcal{F}_{X_1}$*, and* $\Omega_{X_2} \in \mathcal{F}_{X_2}$*,*

$$U\Omega_X = \Omega_{X_1} \otimes \Omega_{X_2}.$$

(ii) *For the decomposition* $h = h_1 \oplus h_2$ *(*$h \in L^2(X)$*,* $h_j \in L^2(X_j)$*,* $j = 1, 2$*),*

$$Ud\Gamma_X(h) = \overline{d\Gamma_{X_1}(h_1) \otimes I + \otimes d\Gamma_{X_2}(h_2)}.$$

Let $\mathcal{V}$ be a separable Hilbert space. Then, for each $n \in \mathbb{N}$ we define the Hilbert space $L^2_{\mathrm{sym}}(X^n; \mathcal{V})$ of all square-integrable, $\mathcal{V}$-valued, symmetric functions:

$$L^2_{\mathrm{sym}}(X^n; \mathcal{V}) \quad := \quad \left\{ f : X^n \to \mathcal{V} \text{ is measurable} \middle| \text{ for each } \sigma \in \mathfrak{S}_n \right.$$

$$f(k_1, \cdots, k_n) = f(k_{\sigma(1)}, \cdots, k_{\sigma(n)})$$

$$\left. \text{and } \int_{X^n} \|f(k_1, \cdots, k_n)\|^2_{\mathcal{V}} d\mu^n(k_1, \cdots, k_n) < \infty \right\},$$

where $\mathfrak{S}_n$ denotes the permutation group of all permutations of $\{1, \cdots, n\}$, i.e., $\mathfrak{S}_n \ni \sigma$ is a bijective map from $\{1, \cdots, n\}$ to itself. We say $f : X^n \to \mathcal{V}$ is measurable if $(v, f(\cdot))_{\mathcal{V}} : X^n \to \mathbb{C}$ is measurable for every $v \in \mathcal{V}$.

The following proposition is well known:

Proposition 2.3 *The two spaces, $\mathcal{V} \otimes \mathcal{F}_X$ and $\bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(X^n; \mathcal{V})$, are unitarily equivalent. Namely, there is a unitary operator $U_{\mathcal{V}} : \mathcal{V} \otimes \mathcal{F}_X \to \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(X^n; \mathcal{V})$ with convention $L^2_{\mathrm{sym}}(X^0; \mathcal{V}) := \mathcal{V}$.*

Through this unitary transformation $U_{\mathcal{V}}$, for every $\Psi \in \mathcal{F}$ we denote $U_{\mathcal{V}} \Psi$ by $\Psi_{\mathcal{V}}$, i.e., $\Psi_{\mathcal{V}} := U_{\mathcal{V}} \Psi$. Moreover, $\Psi_{\mathcal{V}}$ is often expressed as

$$\Psi_{\mathcal{V}} \quad = \quad \oplus \sum_{n=0}^{\infty} \Psi_{\mathcal{V}}^{(n)} = \Psi_{\mathcal{V}}^{(0)} \oplus \Psi_{\mathcal{V}}^{(1)} \oplus \cdots \oplus \Psi_{\mathcal{V}}^{(n)} \oplus \cdots, \qquad (2.1)$$

$$\Psi_{\mathcal{V}}^{(n)} \in L^2_{\mathrm{sym}}(X^n; \mathcal{V}), \quad n \in \{0\} \cup \mathbb{N}.$$

Therefore, the norm $\|\Psi\|_{\mathcal{F}_X}$ has the following expression:

$$\|\Psi\|^2_{\mathcal{F}_X} \quad = \quad \|\Psi_{\mathcal{V}}^{(0)}\|^2_{\mathcal{V}} + \sum_{n=1}^{\infty} \|\Psi_{\mathcal{V}}^{(n)}\|^2_{L^2(X^n; \mathcal{V})}$$

$$= \quad \|\Psi_{\mathcal{V}}^{(0)}\|^2_{\mathcal{V}} + \sum_{n=1}^{\infty} \int_{X^n} \|\Psi_{\mathcal{V}}^{(n)}(k_1, \cdots, k_n)\|^2_{\mathcal{V}} d\mu^n(k_1, \cdots, k_n).$$

Here we give the generalization of [27, Corollary 5.1] together with its proof:

Proposition 2.4 *Let $\{f_\ell\}_{\ell=1}^{\infty}$ be an arbitrary complete orthonormal system of $L^2(X)$. Then,*

$$\|I \otimes N_X^{1/2} \Psi\|^2_{\mathcal{V} \otimes \mathcal{F}_X} = \sum_{\ell=1}^{\infty} \|I \otimes a_X(f_\ell) \Psi\|^2_{\mathcal{V} \otimes \mathcal{F}_X}$$

for every $\Psi \in D(I \otimes N_X^{1/2})$.

Proof. Let $\Psi \in D(I \otimes N_X^{1/2})$. Then, by the definition of the annihilation operator and Proposition 2.3, for each $M \in \mathbb{N}$ we have

$$\sum_{\ell=1}^{M} \|I \otimes a_X(f_\ell)\Psi\|^2_{\mathcal{V} \otimes \mathcal{F}_X} = \sum_{\ell=1}^{M} \|a_{\mathcal{V}}(f_\ell)\Psi_{\mathcal{V}}\|^2_{\oplus_n L^2_{\mathrm{sym}}(X^n;\mathcal{V})}$$

$$= \sum_{n=0}^{\infty}(n+1)\int_{X^n} \Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n)d\mu^n(k_1,\cdots,k_n), \qquad (2.2)$$

where

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) := \sum_{\ell=1}^{M} \left\| \int_X \overline{f_\ell(k)}\, \Psi_{\mathcal{V}}^{(n+1)}(k,k_1,\cdots,k_n)d\mu(k) \right\|^2_{\mathcal{V}}.$$

Let $\{e_p\}_{p=1}^{\infty}$ be an arbitrary complete orthonormal system of $\mathcal{V}$. Then, we have

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n)$$

$$= \sum_{\ell=1}^{M}\sum_{p=1}^{\infty} \left| \left(e_p \,,\, \int_X \overline{f_\ell(k)}\Psi_{\mathcal{V}}^{(n+1)}(k,k_1,\cdots,k_n)d\mu(k) \right)_{\mathcal{V}} \right|^2$$

$$= \sum_{\ell=1}^{M}\sum_{p=1}^{\infty} \left| \int_X \left(f_\ell(k)e_p \,,\, \Psi_{\mathcal{V}}^{(n+1)}(k,k_1,\cdots,k_n) \right)_{\mathcal{V}} d\mu(k) \right|^2.$$

We note here that $\Psi_{\mathcal{V}}^{(n+1)}(\cdot,k_1,\cdots,k_n) \in L^2(X;\mathcal{V})$ for a.e. $(k_1,\cdots,k_n)$. Moreover, $\{f_\ell e_p\}_{\ell,p=1}^{\infty}$ makes a complete orthonormal system of $L^2(X;\mathcal{V})$. Hence it follows from Bessel's inequality that

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) \le \|\Psi_{\mathcal{V}}^{(n+1)}(\cdot,k_1,\cdots,k_n)\|^2_{L^2(X;\mathcal{V})}.$$

Thus, we have

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) \nearrow \int_X \|\Psi_{\mathcal{V}}^{(n+1)}(k,k_1,\cdots,k_n)\|^2_{\mathcal{V}}d\mu(k) \qquad (2.3)$$

as $M \to \infty$. Applying Lebesgue's monotone convergence theorem and Fubini's theorem to Eqs.(2.2) and (2.3), we reach the conclusion:

$$\sum_{\ell=1}^{\infty} \|I \otimes a_X(f_\ell)\Psi\|^2_{\mathcal{V}\otimes\mathcal{F}_X}$$

$$= \sum_{n=0}^{\infty}(n+1)\int_{X^{n+1}} \|\Psi^{(n+1)}(k_1,\cdots,k_{n+1})\|^2_{\mathcal{V}}d\mu^n(k_1,\cdots,k_{n+1})$$

$$= \|I \otimes N_X^{1/2}\Psi\|^2_{\mathcal{V}\otimes\mathcal{F}_X}.$$

$$\square$$

As a special case of Proposition 2.4, namely we only have to take the case where $\mathcal{V} = \mathbb{C}$, we have

Corollary 2.5 *[27, Proposition 5.1] Let $\{f_\ell\}_{\ell=1}^\infty$ be an arbitrary complete orthonormal system of $L^2(X)$. Then,*

$$\|N_X^{1/2}\psi\|_{\mathcal{F}_X}^2 = \sum_{\ell=1}^\infty \|a_X(f_\ell)\psi\|_{\mathcal{F}_X}^2$$

for every $\psi \in D(N_X^{1/2})$.

2.2   The Total Hamiltonian H_{QFT}

Let us give the state space of the physical system represented by a separable, complex Hilbert space $\mathcal{H}$. Only when $X = \mathbb{R}^d$, we use the following abbreviation:

$$\mathcal{F}_{\mathrm{b}} := \mathcal{F}_{\mathbb{R}^d} \equiv \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)).$$

Corresponding to this abbreviation, we abbreviate $a_{\mathbb{R}^d}(f)$, $a_{\mathbb{R}^d}^\dagger(f)$, and $d\Gamma_{\mathbb{R}^d}(h)$ to $a_{\mathrm{b}}(f)$, $a_{\mathrm{b}}^\dagger(f)$, and $d\Gamma_{\mathrm{b}}(h)$, respectively:

$$a_{\mathrm{b}}(f) := a_{\mathbb{R}^d}(f), \quad a_{\mathrm{b}}^\dagger(f) := a_{\mathbb{R}^d}^\dagger(f), \quad d\Gamma_{\mathrm{b}}(h) := d\Gamma_{\mathbb{R}^d}(h).$$

In particular we often use the notation N_{b} for $d\Gamma_{\mathrm{b}}(1)$, i.e.,

$$N_{\mathrm{b}} := d\Gamma_{\mathrm{b}}(1).$$

The total state space of the physical system coupled with the Bose field is given by the tensor product of the two Hilbert spaces:

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}.$$

Let A be a self-adjoint operator acting in $\mathcal{H}$ bounded from below. We suppose the following idealization for the dispersion relation $\omega(k)$ because we are interested in IR behavior around $k = 0$. Let $\omega : \mathbb{R}^d \longrightarrow [0\,,\infty)$ be a continuous function such that $0 < \omega(k) < \infty$ for every $k \in \mathbb{R}^d \setminus \{0\}$ and $\inf_{|k|>\varepsilon} \omega(k) > 0$ for every $\varepsilon > 0$. The unperturbed Hamiltonian of our model is defined by

$$H_0 := A \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega) \tag{2.4}$$

with domain $D(H_0) := D(A \otimes I) \cap D(I \otimes d\Gamma_{\mathrm{b}}(\omega)) \subset \mathcal{F}$, where I denotes identity operator and $D(S)$ the domain of an operator S. The operator H_0 is self-adjoint and bounded from below.

We suppose that our total Hamiltonian has the form:

$$H_{\mathrm{QFT}} = H_0 + H_{\mathrm{I}},$$

and *we always assume H_{QFT} to be a self-adjoint operator acting in $\mathcal{F}$* in this paper and we suppose it to describe our model of the physical system coupled with the quantum field. Here H_{I} is the interaction Hamiltonian.

Let $\ker(S)$ stand for the kernel of an operator S, i.e.,

$$\ker(S) := \{\Psi \in \mathcal{F} \,|\, S\Psi = 0\}.$$

In addition, when S is closed, let us denote by $\sigma(S)$ the spectrum of a closed operator S.

Definition 2.6 By *ground state energy* we mean $\inf \sigma(H_{\mathrm{QFT}})$, the lowest spectrum of H_{QFT}. We denote the ground state energy by $E_0(H_{\mathrm{QFT}})$, i.e., $E_0(H_{\mathrm{QFT}}) := \inf \sigma(H_{\mathrm{QFT}})$. We say H_{QFT} *has a ground state* Ψ_{QFT} if $\ker(H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}))$ is not empty and then $0 \neq \Psi_{\mathrm{QFT}} \in \ker(H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}))$. We say Ψ_{QFT} to be normalized if $\|\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} = 1$.

For simplicity, we set $\widehat{H}_{\mathrm{QFT}}$ as

$$\widehat{H}_{\mathrm{QFT}} := H_{\mathrm{QFT}} - E_0(H_{\mathrm{QFT}}).$$

We always suppose that Ψ_{QFT} has been normalized whenever it exists.

3 Domain Properties of the Carleman Operator for IR Catastrophe

When the operator-theoretical pull-through (OPPT) formula on ground states holds in the same way as in [27], $a(f)\Psi_{\mathrm{QFT}}$ has the expression:

$$a(f)\Psi_{\mathrm{QFT}} = -\int_{\mathbb{R}^d} \overline{f(k)} \left(\widehat{H}_{\mathrm{QFT}} + \omega(k) \right)^{-1} B_{\mathrm{PT}}(k) \Psi_{\mathrm{QFT}} dk \qquad (3.1)$$

for every $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. This is the operator-theoretical version of the symbolical pull-through formula on ground states:

$$a(k)\Psi_{\mathrm{QFT}} = -\left(\widehat{H}_{\mathrm{QFT}} + \omega(k) \right)^{-1} B_{\mathrm{PT}}(k) \Psi_{\mathrm{QFT}}. \qquad (3.2)$$

Then, we have an operator $B_{\mathrm{PT}}(k)$ for every $k \in \mathbb{R}^d \setminus \{0\}$ in the integrand of Eq.(3.1). We can show OPPT formula holds for several models in quantum field theory [6, 27, 33].

In our argument, we assume the following conditions:

(Ass.1) Eq. (3.1) holds and then $B_{\mathrm{PT}}(k)$ is determined for every $k \in \mathbb{R}^d \setminus \{0\}$ as an operator acting in $\mathcal{F}$ and then $B_{\mathrm{PT}}(\cdot)\Psi$ is measurable for every $\Psi \in D(H_0)$ (i.e., $D(B_{\mathrm{PT}}(k)) \supset D(H_0)$ for every $k \in \mathbb{R}^d \setminus \{0\}$ and $(\Phi, B_{\mathrm{PT}}(\cdot)\Psi)_{\mathcal{F}} : \mathbb{R}^d \longrightarrow \mathbb{C}$ is measurable for every $\Phi \in \mathcal{F}$).

(Ass.2) $(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)$ is bounded for every $k \in \mathbb{R}^d \setminus \{0\}$ and then for every $\varepsilon > 0$

$$M_\varepsilon := \left\{ \int_{|k|>\varepsilon} \|(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)\|_{\mathcal{B}(\mathcal{F})}^2 dk \right\}^{1/2} < \infty, \qquad (3.3)$$

where $\|\cdot\|_{\mathcal{B}(\mathcal{F})}$ denotes the operator norm of $\mathcal{B}(\mathcal{F})$, the C^*-algebra of bounded operators on $\mathcal{F}$.

For every $\varepsilon \geq 0$, we set $\mathbb{R}_{<\varepsilon}^d$ and $\mathbb{R}_{>\varepsilon}^d$ as

$$\mathbb{R}_{<\varepsilon}^d := \left\{ k \in \mathbb{R}^d \,\middle|\, |k| < \varepsilon \right\} \quad \text{and} \quad \mathbb{R}_{>\varepsilon}^d := \left\{ k \in \mathbb{R}^d \,\middle|\, |k| > \varepsilon \right\},$$

respectively. For every $f \in L^2(\mathbb{R}^d)$ we define $f^{<\varepsilon}$ and $f^{>\varepsilon}$ in $L^2(\mathbb{R}^d)$ by

$$f^{<\varepsilon}(k) := 1^{<\varepsilon}(k)f(k) \quad \text{and} \quad f^{>\varepsilon}(k) := 1^{>\varepsilon}(k)f(k),$$

where $1^{<\varepsilon}$ and $1^{>\varepsilon}$ are characteristic functions defined by

$$1^{<\varepsilon}(k) := \begin{cases} 1 & \text{if } |k| < \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad 1^{>\varepsilon}(k) := \begin{cases} 1 & \text{if } |k| > \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Since we can regard $f^{<\varepsilon}$ and $f^{>\varepsilon}$ as functions in $L^2(\mathbb{R}^d_{<\varepsilon})$ and $L^2(\mathbb{R}^d_{>\varepsilon})$ respectively, we often handle them as $f^{<\varepsilon} \in L^2(\mathbb{R}^d_{<\varepsilon})$ and $f^{>\varepsilon} \in L^2(\mathbb{R}^d_{>\varepsilon})$ in this paper.

Following this decomposition, we introduce some abbreviations:

$$d\Gamma_{<\varepsilon}(h^{<\varepsilon}) := d\Gamma_{\mathbb{R}^d_{<\varepsilon}}(h^{<\varepsilon}), \qquad d\Gamma_{>\varepsilon}(h^{>\varepsilon}) := d\Gamma_{\mathbb{R}^d_{>\varepsilon}}(h^{>\varepsilon}),$$

$$a^\sharp_{<\varepsilon}(f^{<\varepsilon}) := a^\sharp_{\mathbb{R}^d_{<\varepsilon}}(f^{<\varepsilon}), \qquad a^\sharp_{>\varepsilon}(f^{>\varepsilon}) := a^\sharp_{\mathbb{R}^d_{>\varepsilon}}(f^{>\varepsilon}),$$

$$\text{and} \qquad a^\sharp(f) := I \otimes a^\sharp_{\mathrm{b}}(f) = I \otimes a^\sharp_{\mathbb{R}^d}(f),$$

where $a^\sharp_X$ denotes a_X or $a^\dagger_X$.

By Proposition 2.2, there exists a unitary operator U_ε for every $\varepsilon > 0$ such that

$$U_\varepsilon \mathcal{F} = \mathcal{H} \otimes \mathcal{F}_{\mathbb{R}^d_{<\varepsilon}} \otimes \mathcal{F}_{\mathbb{R}^d_{>\varepsilon}} \equiv \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_{<\varepsilon})) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_{>\varepsilon})) =: \mathcal{F}_\varepsilon.$$

Write $U_\varepsilon \Psi \in \mathcal{F}_\varepsilon$ as Ψ_ε for every $\Psi \in \mathcal{F}$, i.e., $\Psi_\varepsilon := U_\varepsilon \Psi$. Then, Proposition 2.2(ii) leads us to the relation:

$$U_\varepsilon(I \otimes d\Gamma_{\mathrm{b}}(h))U_\varepsilon^* = \overline{I \otimes d\Gamma_{<\varepsilon}(h^{<\varepsilon}) \otimes I + I \otimes I \otimes d\Gamma_{>\varepsilon}(h^{>\varepsilon})} \quad (3.4)$$

for every real-valued function $h : \mathbb{R}^d \to \mathbb{R}$.

We define the *boson number operator* N acting in $\mathcal{F}$ by

$$N := I \otimes d\Gamma_{\mathrm{b}}(1). \tag{3.5}$$

Symbolically we set the ground-state expectation $\langle S \rangle_{\mathrm{gs}}$ for an operator S acting in $\mathcal{F}$ by $\langle S \rangle_{\mathrm{gs}} := (\Psi_{\mathrm{QFT}}, S\Psi_{\mathrm{QFT}})_\mathcal{F}$. Here we note Ψ_{QFT} is normalized, i.e., $\|\Psi_{\mathrm{QFT}}\|_\mathcal{F} = 1$. Then, we can consider $\langle S \rangle_{\mathrm{gs}}$ to be finite if $\Psi_{\mathrm{QFT}} \in D(S)$, on the other hand, to be infinite if $\Psi_{\mathrm{QFT}} \notin D(S)$. We note we can write $\Psi_{\mathrm{QFT}} \notin D(S)$ when Ψ_{QFT} does not exist in $\mathcal{F}$. That is,

$$\langle S \rangle_{\mathrm{gs}} < \infty \text{ if } \Psi_{\mathrm{QFT}} \in D(S),$$
$$\langle S \rangle_{\mathrm{gs}} = \infty \text{ if } \Psi_{\mathrm{QFT}} \notin D(S) \text{ or } \Psi_{\mathrm{QFT}} \text{ does not exist in } \mathcal{F}.$$

Definition 3.1 We say the *infrared (IR) catastrophe occurs* if $\Psi_{\mathrm{QFT}} \notin D(N^{1/2})$ including the case where Ψ_{QFT} does not exist in $\mathcal{F}$, i.e., $\Psi_{\mathrm{QFT}} \notin \mathcal{F}$.

Remark 3.1 *Since* $D(N) \subset D(N^{1/2})$, *the naive meaning of Definition 3.1 is symbolically:*

$$\langle N \rangle_{\mathrm{gs}} = (\Psi_{\mathrm{QFT}}, N\Psi_{\mathrm{QFT}})_\mathcal{F} = \|N^{1/2}\Psi_{\mathrm{QFT}}\|_\mathcal{F}^2 = \infty.$$

For every $\varepsilon > 0$, we define $N_{>\varepsilon}$ acting in $\mathcal{F}$ by

$$N_{>\varepsilon} := U_\varepsilon^*(I \otimes I \otimes d\Gamma_{>\varepsilon}(1^{>\varepsilon}))U_\varepsilon.$$

Lemma 3.2 *[21]*

$$D(H_0) \subset \bigcap_{\varepsilon > 0} D(N_{>\varepsilon}^{1/2}).$$

Proof . Since $1^{>\varepsilon}(k) \leq (\inf_{|k|>\varepsilon} \omega(k))^{-1}\omega^{>\varepsilon}(k)$ for every $\varepsilon > 0$, we have $D(\omega^{>\varepsilon}) \subset D(1^{>\varepsilon})$, which implies $D(I \otimes d\Gamma_{>\varepsilon}(\omega^{>\varepsilon})) \subset D(I \otimes d\Gamma_{>\varepsilon}(1^{>\varepsilon}))$. Thus, by Eq.(3.4) we have

$$
\begin{aligned}
D(H_0) &= D(A \otimes I) \cap D(I \otimes d\Gamma(\omega)) \\
&\cong D(A \otimes I \otimes I) \cap D(I \otimes d\Gamma(\omega^{<\varepsilon}) \otimes I) \cap D(I \otimes I \otimes d\Gamma(\omega^{>\varepsilon})) \\
&\subset \mathcal{F}_\varepsilon \cap D(I \otimes I \otimes d\Gamma(1^{>\varepsilon})) = D(I \otimes I \otimes d\Gamma(1^{>\varepsilon})) \cong D(N_{>\varepsilon}).
\end{aligned}
$$

Combining the fact that $D(N_{>\varepsilon}) \subset D(N_{>\varepsilon}^{1/2})$ with the above leads us to our lemma. $\square$

To find a relation between N and $N_{>\varepsilon}$, we introduce the following domain:

$$D_{\mathrm{CNB}} := \left\{ \Psi \in \bigcap_{\varepsilon > 0} D(N_{>\varepsilon}^{1/2}) \,\middle|\, \sup_{\varepsilon > 0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 < \infty \right\}.$$

The following lemma is a mathematical establishment of Eq.(1.2).

Lemma 3.3 *Let $\{f_\ell^{>\varepsilon}\}_{\ell=1}^\infty$ be an arbitrary complete orthonormal system of $L^2(\mathbb{R}_{>\varepsilon}^d)$ for every $\varepsilon > 0$. Then,*

$$\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \sum_{\ell=0}^\infty \|a(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 = \sum_{\ell=0}^\infty \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2$$

for every $\Psi \in D(N_{>\varepsilon}^{1/2})$.

Proof. By Proposition 2.4, we have

$$\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \|I \otimes I \otimes d\Gamma(1_{>\varepsilon})^{1/2}\Psi_\varepsilon\|_{\mathcal{F}_\varepsilon}^2 = \sum_{\ell=1}^\infty \|I \otimes I \otimes a_{>\varepsilon}(f_\ell^{>\varepsilon})\Psi_\varepsilon\|_{\mathcal{F}_\varepsilon}^2$$

$$= \sum_{\ell=1}^\infty \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2.$$

The above equation, together with $\|a(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 = \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 = \|I \otimes I \otimes a_{>\varepsilon}(f_\ell^{>\varepsilon})\Psi_\varepsilon\|_{\mathcal{F}_\varepsilon}^2$, completes the proof of our lemma. $\square$

The following lemma gives a relation between N and $N_{>\varepsilon}$. It tells us that for all vectors $\Psi \in D(H_0)$ we can check whether $(\Psi, N\Psi)_{\mathcal{F}}$ converges by taking advantage of Lemma 3.2 and estimating $\sup_{\varepsilon > 0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2$. Thus, the following lemma plays an important role to prove Theorems 3.6 and 3.7 below.

Lemma 3.4 $D_{\mathrm{CNB}} = D(N^{1/2})$ *and*

$$\sup_{\varepsilon>0}\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2$$

for $\Psi \in D_{\mathrm{CNB}}$.

Proof. Let $\Psi \in D(N^{1/2})$ first. Then, we can show $\Psi \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$ in the same way as the proof of Lemma 3.2. Let $\{f_\ell\}_{\ell=1}^\infty$ be an arbitrary complete orthonormal system of $L^2(\mathbb{R}^d)$. We decompose f_ℓ into $f_\ell^{<\varepsilon}$ and $f_\ell^{>\varepsilon}$, i.e., $f_\ell = f_\ell^{<\varepsilon} + f_\ell^{>\varepsilon}$. Then, evidently $\{f_\ell^{>\varepsilon}\}_{\ell=1}^\infty$ makes a complete orthonormal system of $L^2(\mathbb{R}_{>\varepsilon}^d)$ and moreover $f_\ell^{>\varepsilon} \to f_\ell$ in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$.

To obtain $\sup_{\varepsilon>0} \sum_{\ell=1}^\infty \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2$ we carefully revise the method which we used in the proof of Proposition 2.4. For each $M \in \mathbb{N}$ we can derive

$$\sum_{\ell=1}^M \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 = \sum_{n=0}^\infty (n+1) \int_{\mathbb{R}^{dn}} \Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) dk_1 \cdots dk_n \quad (3.6)$$

from the definition of the annihilation operator, where we used the representation (2.1) and

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) := \sum_{\ell=0}^M \left\| \int_{|k|>\varepsilon} \overline{f_\ell^{>\varepsilon}(k)}\, \Psi_{\mathcal{H}}^{(n+1)}(k,k_1,\cdots,k_n) dk \right\|_{\mathcal{H}}^2.$$

Let $\{e_p\}_{p=1}^\infty$ be an arbitrary complete orthonormal system of $\mathcal{H}$. Then, we have

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) = \sum_{\ell=1}^M \sum_{p=1}^\infty \left| \int_{\mathbb{R}_{>\varepsilon}^d} \left(f_\ell^{>\varepsilon}(k)e_p\,,\, \Psi_{\mathcal{H}}^{(n+1)}(k,k_1,\cdots,k_n) \right)_{\mathcal{H}} dk \right|^2.$$

We note here that we can regard $\Psi_{\mathcal{H}}^{(n+1)}(\cdot,k_1,\cdots,k_n)$ as a function in $L^2(\mathbb{R}_{>\varepsilon}^d; \mathcal{H})$ for a.e. $(k_1,\cdots,k_n)$ because $\Psi_{\mathcal{H}}^{(n+1)}(\cdot,k_1,\cdots,k_n) \in L^2(\mathbb{R}^d; \mathcal{H})$ for a.e. $(k_1,\cdots,k_n)$. Moreover, $\{f_\ell^{>\varepsilon}e_p\}_{\ell,p=1}^\infty$ makes a complete orthonormal system of $L^2(\mathbb{R}_{>\varepsilon}^d; \mathcal{H})$. Hence it follows from Bessel's inequality that

$$
\begin{aligned}
\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) &\leq \|\Psi_{\mathcal{H}}^{(n+1)}(\cdot,k_1,\cdots,k_n)\|_{L^2(\mathbb{R}_{>\varepsilon}^d;\mathcal{H})}^2 \\
&\leq \|\Psi_{\mathcal{H}}^{(n+1)}(\cdot,k_1,\cdots,k_n)\|_{L^2(\mathbb{R}^d;\mathcal{H})}^2.
\end{aligned}
$$

Thus, we have

$$\Psi_{M,\varepsilon}^{(n)}(k_1,\cdots,k_n) \nearrow \int_{|k|>\varepsilon} \|\Psi_{\mathcal{H}}^{(n+1)}(k,k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk \quad (3.7)$$

as $M \to \infty$. Applying Lebesgue's monotone convergence theorem and Fubini's theorem to Eqs.(3.6) and (3.7), we reach the expression:

$$
\begin{aligned}
&\sum_{\ell=1}^\infty \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 \\
&= \sum_{n=0}^\infty (n+1) \int_{|k|>\varepsilon} \int_{\mathbb{R}^{dn}} \|\Psi^{(n+1)}(k,k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1 \cdots dk_n dk.
\end{aligned}
$$

From this expression, we know that $\sum_{\ell=1}^{\infty} \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2$ is increasing as $\varepsilon \to 0$. So, applying Lebesgue's monotone convergence theorem to the above equation yields

$$
\begin{aligned}
\sup_{\varepsilon>0} \sum_{\ell=1}^{\infty} \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 &= \lim_{\varepsilon\to 0} \sum_{\ell=1}^{\infty} \|I \otimes a_{\mathrm{b}}(f_\ell^{>\varepsilon})\Psi\|_{\mathcal{F}}^2 \\
&= \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{d(n+1)}} \|\Psi^{(n+1)}(k_1,\cdots,k_{n+1})\|_{\mathcal{H}}^2 dk_1\cdots dk_{n+1} \\
&= \|N^{1/2}\Psi_{\mathcal{H}}\|_{\bigoplus_n L^2_{\mathrm{sym}}(\mathbb{R}^{dn};\mathcal{H})}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2 . \qquad (3.8)
\end{aligned}
$$

By Lemma 3.3 and Eq. (3.8), we obtain that

$$
\sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 = \|N^{1/2}\Psi\|_{\mathcal{F}}^2 < \infty, \qquad (3.9)
$$

which means $\Psi \in D_{\mathrm{CNB}}$.

Conversely, let $\Psi = \oplus \sum_{n=0}^{\infty} \Psi^{(n)} \in D_{\mathrm{CNB}}$. Using symmetry of $\Psi^{(n)}(k_1,\cdots,k_n)$, we have

$$
\begin{aligned}
\|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 &= \sum_{n=1}^{\infty} \|I \otimes I \otimes d\Gamma_{>\varepsilon}(1_{>\varepsilon})^{1/2} \Psi_\varepsilon^{(n)}\|_{\mathcal{F}_\varepsilon}^2 \\
&= \sum_{n=1}^{\infty} \Bigg\{ n \int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \\
&\quad + \sum_{j=1}^{n-1} \binom{n}{j} j \int_{|k_1|,\cdots,|k_j|>\varepsilon\,;\,|k_{j+1}|,\cdots,|k_n|<\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \\
&\quad + \int_{|k_1|,\cdots,|k_n|<\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n \Bigg\} \\
&\geq \sum_{n=1}^{\infty} n \int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n =: \Theta(\varepsilon).
\end{aligned}
$$

Here let us set $A_\nu(\varepsilon)$ as

$$
A_\nu(\varepsilon) := \sum_{n=1}^{\nu} n \int_{|k_1|,\cdots,|k_n|>\varepsilon} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n .
$$

Then, Lebesgue's monotone convergence theorem implies

$$
A_\nu(\varepsilon) \longrightarrow \sum_{n=1}^{\nu} n \int_{\mathbb{R}^{dn}} \|\Psi^{(n)}(k_1,\cdots,k_n)\|_{\mathcal{H}}^2 dk_1\cdots dk_n =: A_\nu \qquad (3.10)
$$

as $\varepsilon \to 0$. Since $\Theta(\varepsilon)$ is increasing as ε tends to 0, we have

$$
\infty > \sup_{\varepsilon>0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 \geq \lim_{\varepsilon\to 0} \Theta(\varepsilon) \geq \lim_{\varepsilon\to 0} A_\nu(\varepsilon) = A_\nu \qquad (3.11)
$$

for each $\nu \in \mathbb{N}$. By Eqs.(3.10) and (3.11), $\{A_\nu\}_{\nu \in \mathbb{N}}$ is monotone increasing and bounded. Therefore, $\lim_{\nu \to \infty} A_\nu$ exists and then we have

$$\infty > \lim_{\nu \to \infty} A_\nu = \sum_{n=1}^{\infty} n \int_{\mathbb{R}^{dn}} \|\Psi^{(n)}(k_1, \cdots, k_n)\|_{\mathcal{H}}^2 dk_1 \cdots dk_n = \|N^{1/2}\Psi\|_{\mathcal{F}}^2.$$

Hence it follows from this and Eq.(3.11) that

$$\infty > \sup_{\varepsilon > 0} \|N_{>\varepsilon}^{1/2}\Psi\|_{\mathcal{F}}^2 \geq \|N^{1/2}\Psi\|_{\mathcal{F}}^2. \qquad (3.12)$$

So, we reach the conclusion that $\Psi \in D(N^{1/2})$, and thus, Eq.(3.9) holds. $\square$

In [28], the author tried to use Fatou's lemma to prove Eq.(3.12). But there was a mistake in his proof. In [11], Bruneau completes the author's idea. Namely, we have

$$\|N_{>\varepsilon}^{1/2}\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} j \|\underbrace{(\overline{p}_\varepsilon \otimes \cdots \otimes \overline{p}_\varepsilon)}_{n-j} \otimes \underbrace{(p_\varepsilon \otimes \cdots \otimes p_\varepsilon)}_{j} \Psi^{(n)}\|_{\mathcal{F}}^2,$$

where p_ε is the orthogonal projection from $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}_{>\varepsilon}^d)$ and $\overline{p}_\varepsilon := 1 - p_\varepsilon$. So, applying Fatou's lemma, we have also Eq.(3.12).

Definition 3.5 When a ground state Ψ_{QFT} of H_{QFT} exists, we can define an $\mathcal{F}$-valued function $K_{\mathrm{PT}} : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathcal{F}$ by

$$K_{\mathrm{PT}}(k) := (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}} \qquad (3.13)$$

for every $k \in \mathbb{R}^d \setminus \{0\}$ since $(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)$ is a bounded operator on $\mathcal{F}$ for every $k \in \mathbb{R}^d \setminus \{0\}$ by (Ass.2). K_{PT} defined by Eq.(3.13) is measurable by (Ass.1). For the ground state Ψ_{QFT}, we define the *maximal Carleman operator* $T_{\mathrm{PT}} : \mathcal{F} \to L^2(\mathbb{R}^d)$ induced by K_{PT} in the following:

$$D(T_{\mathrm{PT}}) := \left\{ \Phi \in \mathcal{F} \,\Big|\, (K_{\mathrm{PT}}(\cdot)\,, \Phi)_{\mathcal{F}} \in L^2(\mathbb{R}^d) \right\}, \qquad (3.14)$$
$$(T_{\mathrm{PT}}\Phi)(k) := (K_{\mathrm{PT}}(k)\,, \Phi)_{\mathcal{F}}\,, \quad \Phi \in D(T_{\mathrm{PT}}),$$

for every $k \in \mathbb{R}^d \setminus \{0\}$. Then, we call K_{PT} the *inducing function* of T_{PT}. When K_{PT} has a singularity at $k = 0$, we call it *IR singularity of T_{PT}*.

We note that T_{PT} is closed by [56, Theorem 6.13].

The following theorem is stated in [33, Theorem 2.9] by Hiroshima. We now give it another proof by obeying the Dereziński-Gérard idea [21] and taking advantage of Lemma 3.4.

Theorem 3.6 *Assume $D(H_{\mathrm{QFT}}) = D(H_0)$ and there exists a ground state Ψ_{QFT} of H_{QFT}. Then, the following conditions are equivalent:*

(i) $\Psi_{\mathrm{QFT}} \in D(N^{1/2})$.

(ii) $\|K_{\mathrm{PT}}(\cdot)\|_{\mathcal{F}} \in L^2(\mathbb{R}^d)$.

(iii) T_{PT} *is a Hilbert-Schmidt operator.*

If one of them holds, then $\|T_{\mathrm{PT}}\| \leq M_0 := \lim_{\varepsilon \to 0} M_\varepsilon < \infty.$

Proof . Before proving this theorem, we first show Eq.(3.18) below. Let $\left\{ f_\ell^{>\varepsilon} \right\}_{\ell=1}^\infty$ be an arbitrary complete orthonormal system of $L^2(\mathbb{R}_{>\varepsilon}^d)$. Lemma 3.2 leads us to the fact that $\Psi_{\mathrm{QFT}} \in \bigcap_{\varepsilon>0} D(N_{>\varepsilon}^{1/2})$. By Lemma 3.3, we have

$$\infty > \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \sum_{\ell=1}^\infty \|a(f_\ell^{>\varepsilon})\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2, \qquad (3.15)$$

Here, in the same way as for T_{PT}, we define the maximal Carleman operator $T_\varepsilon : \mathcal{F} \to L^2(\mathbb{R}_{>\varepsilon}^d)$ by

$$D(T_\varepsilon) := \left\{ \Phi \in \mathcal{F} \,\middle|\, (K_{\mathrm{PT}}(\cdot)\,,\,\Phi)_{\mathcal{F}} \in L^2(\mathbb{R}_{>\varepsilon}^d) \right\},$$
$$(T_\varepsilon \Phi)(k) := (K_{\mathrm{PT}}(k)\,,\,\Phi)_{\mathcal{F}}\,,\quad \Phi \in D(T_\varepsilon), \qquad (3.16)$$

for every $k \in \mathbb{R}_{>\varepsilon}^d$. By the condition (3.3), T_ε is a bounded operator with $D(T_\varepsilon) = \mathcal{F}$ and $\|T_\varepsilon\| \leq M_\varepsilon$. Thus, the adjoint operator T_ε^* of T_ε is well-defined. Here, remember that we employed the normalized ground state Ψ_{QFT} if it exists. By Eqs.(3.1), (3.13), and (3.16), we have

$$\left(\Phi\,,\,I \otimes a(f_\ell^{>\varepsilon})\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} = -\overline{\left(\overline{f_\ell^{>\varepsilon}}\,,\,T_\varepsilon \Phi\right)}_{\mathcal{F}} = -\left(\Phi\,,\,T_\varepsilon^* \overline{f_\ell^{>\varepsilon}}\right)_{\mathcal{F}}$$

for every $\Phi \in \mathcal{F}$, which implies

$$a(f_\ell^{>\varepsilon})\Psi_{\mathrm{QFT}} = -T_\varepsilon^* \overline{f_\ell^{>\varepsilon}} \qquad (3.17)$$

for each $\ell \in \mathbb{N}$. Thus, by applying [47, Theorems VI.18] to Eqs.(3.15) and (3.17), we have

$$\infty > \|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 \;\;=\;\; \sum_{\nu=0}^\infty \|T_\varepsilon^* \overline{f}_\nu\|_{\mathcal{F}}^2 = \sum_{\nu=0}^\infty \left(\overline{f}_\nu\,,\,T_\varepsilon T_\varepsilon^* \overline{f}_\nu\right)_{L^2(\mathbb{R}_{>\varepsilon}^d)} = \mathrm{tr}(T_\varepsilon T_\varepsilon^*).$$

Thus, [47, VI.22] tells us that T_ε^* is a Hilbert-Schmidt operator and thus T_ε is also a Hilbert-Schmidt operator. Thus, [56, Theorems 6.12, 6.13] leads us to the fact that $\|K_{\mathrm{PT}}(\cdot)\|_{\mathcal{F}} \in L^2(\mathbb{R}_{>\varepsilon}^d)$. Let $\{\Phi_p\}_{p=1}^\infty$ be an arbitrary complete orthonormal system of $\mathcal{F}$. Then, by Lemma 3.3, and Eqs.(3.1) and (3.13), we have

$$\|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{PT}}\|_{\mathcal{F}}^2 = \sum_{\ell=1}^\infty \|I \otimes a(f_\ell^{>\varepsilon})\Psi_{\mathrm{PT}}\|_{\mathcal{F}}^2 = \sum_{\ell=1}^\infty \left\| \left(f_\ell^{>\varepsilon}\,,\,K_{\mathrm{PT}}\right)_{L^2(\mathbb{R}_{>\varepsilon}^d)} \right\|_{\mathcal{F}}^2$$
$$= \;\; \sum_{\ell=1}^\infty \sum_{p=1}^\infty \left| \int_{|k|>\varepsilon} \left(f_\ell^{>\varepsilon}(k)\Phi_p\,,\,K_{\mathrm{PT}}(k)dk\right)_{\mathcal{F}} \right|^2.$$

Since $\{f_\ell^{>\varepsilon}\Phi_p\}_{\ell,p=0}^\infty$ makes a complete orthonormal system of $L^2(\mathbb{R}_{>\varepsilon}^d;\mathcal{F})$, this equation yields that $\|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \|K_{\mathrm{PT}}(\cdot)\|_{L^2(\mathbb{R}_{>\varepsilon}^d;\mathcal{F})}^2$. Namely,

$$\|N_{>\varepsilon}^{1/2}\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}^2 = \int_{|k|>\varepsilon} \|K_{\mathrm{PT}}(k)\|_{\mathcal{F}}^2 dk < \infty. \qquad (3.18)$$

We start by showing the equivalence of (i) and (ii). It follows immediately from Lebesgue's monotone convergence theorem, Lemma 3.4, and Eq.(3.18).

We proceed to the the equivalence of (ii) and (iii). It follows directly from [56, Theorem 6.12] that (iii) implies (ii). Thus, conversely, we assume $\|K_{\mathrm{PT}}(\cdot)\|_{\mathcal{F}} \in L^2(\mathbb{R}^d)$ now. Then, for every $\Phi \in \mathcal{F}$ we have

$$\int_{\mathbb{R}^d} \big| \left(K_{\mathrm{PT}}(k)\,,\, \Phi \right)_{\mathcal{F}} \big|^2 dk \leq \int_{\mathbb{R}^d} \|K_{\mathrm{PT}}(k)\|_{\mathcal{F}}^2 dk \, \|\Phi\|_{\mathcal{F}}^2 < \infty$$

by Schwarz's inequality, which implies that T_{PT} is a bounded operator with $D(T_{\mathrm{PT}}) = \mathcal{F}$ and $\|T_{\mathrm{PT}}\| \leq M_0$. Obeying [56, Theorem 6.12], T_{PT} is the restriction of a Hilbert-Schmidt operator since $\|K_{\mathrm{PT}}(\cdot)\|_{\mathcal{F}} \in L^2(\mathbb{R}^d)$. Therefore, T_{PT} itself is a Hilbert-Schmidt operator. $\qquad\square$

Theorem 3.6 tells us that $D(T_{\mathrm{PT}}) = \mathcal{F}$ if Ψ_{QFT} exists in $D(N^{1/2})$. Thus, it is trivial that $D(N^{1/2}) \subset D(T_{\mathrm{PT}})$ in this case. But, more generally, we have this relation in the following theorem, even though Ψ_{QFT} exists outside $D(N^{1/2})$. This theorem is proved by taking advantage of Lemma 3.4:

Theorem 3.7 *Suppose that $D(H_{\mathrm{QFT}}) = D(H_0)$. If a ground state Ψ_{QFT} of H_{QFT} exists, then*

$$D(T_{\mathrm{PT}}) \supset D(N^{1/2}).$$

Proof . Let $\Phi \in D(N^{1/2})$. Then, $\Phi \in D_{\mathrm{CNB}}$ by Lemma 3.4. We define a functional $F_{\Phi} : C_0^{\infty}(\mathbb{R}_{>\varepsilon}^d) \to \mathbb{C}$ by $F_{\Phi}(f) := (a(f)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}}$ for every $f \in C_0^{\infty}(\mathbb{R}_{>\varepsilon}^d)$. Since $\Phi \in D(N_{>\varepsilon}^{1/2})$ for every $\varepsilon > 0$ and Ψ_{QFT} is normalized, Proposition 2.1 leads us to the inequality:

$$|F_{\Phi}(f)| \leq \|f\|_{L^2(\mathbb{R}_{>\varepsilon}^d)} \|(N_{>\varepsilon} + I)^{1/2}\Phi\|_{\mathcal{F}}.$$

Since $C_0^{\infty}(\mathbb{R}_{>\varepsilon}^d)$ is dense in $L^2(\mathbb{R}_{>\varepsilon}^d)$, we have a bounded functional from $L^2(\mathbb{R}_{>\varepsilon}^d)$ to $\mathbb{C}$ as the extension of F_{Φ}. We denote it by the same symbol, i.e., $F_{\Phi}(f) : L^2(\mathbb{R}_{>\varepsilon}^d) \to \mathbb{C}$. By Riesz's lemma, there exists $u_{\Phi} \in L^2(\mathbb{R}_{>\varepsilon}^d)$ such that

$$F_{\Phi}(f) = (u_{\Phi}\,,\,f)_{L^2(\mathbb{R}_{>\varepsilon}^d)}\,,$$

$$\|u_{\Phi}\|_{L^2(\mathbb{R}_{>\varepsilon}^d)} = \|F_{\Phi}\|_{L^2(\mathbb{R}_{>\varepsilon}^d)^*} \leq \|(N_{>\varepsilon} + I)^{1/2}\Phi\|_{\mathcal{F}}, \qquad (3.19)$$

where $L^2(\mathbb{R}_{>\varepsilon}^d)^*$ is the dual space of $L^2(\mathbb{R}_{>\varepsilon}^d)$. It follows from (Ass.1) that for $f \in C_0^{\infty}(\mathbb{R}_{>\varepsilon}^d)$

$$(f\,,\,u_{\Phi})_{L^2(\mathbb{R}_{>\varepsilon}^d)} = \overline{F_{\Phi}(f)} = (\Phi\,,\,a(f)\Psi_{\mathrm{QFT}})_{\mathcal{F}}$$

$$= -\int_{|k|>\varepsilon} \overline{f(k)} \left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} dk.$$

Since $C_0^{\infty}(\mathbb{R}_{>\varepsilon}^d)$ is dense in $L^2(\mathbb{R}_{>\varepsilon}^d)$,

$$L^2(\mathbb{R}_{>\varepsilon}^d) \ni u_{\Phi} = -\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}} + \omega(\cdot))^{-1} B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}} \right)_{\mathcal{F}}.$$

By this and the inequality (3.19), we have

$$\int_{|k|>\varepsilon}\left|\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}}\right|^2 dk$$

$$= \quad \|u_\Phi\|^2_{L^2(\mathbb{R}^d_{>\varepsilon})} \le \|(N_{>\varepsilon}+I)^{1/2}\Phi\|^2_{\mathcal{F}}$$

$$\le \quad \sup_{\varepsilon>0}\|N^{1/2}_{>\varepsilon}\Phi\|^{1/2}_{\mathcal{F}} + \|\Phi\|^{1/2}_{\mathcal{F}} < \infty,$$

since $\Phi \in D_{\mathrm{CNB}}$. So, Lebesgue's monotone convergence theorem gives the following estimate:

$$\int_{\mathbb{R}^d}\left|\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}}\right|^2 dk$$

$$= \quad \lim_{\varepsilon\to 0}\int_{|k|>\varepsilon}\left|\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(k))^{-1}B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}}\right|^2 dk$$

$$< \quad \sup_{\varepsilon>0}\|N^{1/2}_{>\varepsilon}\Phi\|^{1/2}_{\mathcal{F}} + \|\Phi\|^{1/2}_{\mathcal{F}} < \infty.$$

Thus, $(K_{\mathrm{PT}}(\cdot)\,,\,\Phi) \in L^2(\mathbb{R}^d)$, which means $\Phi \in D(T_{\mathrm{PT}})$. $\square$

Theorem 3.7 gives a sufficient condition of IR catastrophe:

Corollary 3.8 *Suppose that* $D(H_{\mathrm{QFT}}) = D(H_0)$. *If* $\Psi_{\mathrm{QFT}} \notin D(T_{\mathrm{PT}})$, *then IR catastrophe occurs.*

Thus, next problem is when Ψ_{QFT} is not in $D(T_{\mathrm{PT}})$ if Ψ_{QFT} exists. Theorem 4.5 below deals with this question in the case where IR singularity of T_{PT} is determined by the singularity of a function λ on $\mathbb{R}^d$. To prove Theorem 4.5, we need the following property of the domain of the Carleman operator:

Theorem 3.9 *Suppose* $D(H_{\mathrm{QFT}}) = D(H_0)$. *Assume a function* λ *on* $\mathbb{R}^d$ *represents IR singularity of* T_{PT} *as the following (1)–(3):*

(1) *there is an* $\varepsilon_0 > 0$ *such that* $\lambda(k) \neq 0$ *for every* $k \in \mathbb{R}^d$ *with* $0 < |k| < \varepsilon_0$,

(2) $\lambda/\omega \notin L^2(K)$ *for every neighborhood* K *of* $k = 0$,

(3) *there is an operator* $B_0(0)$ *acting in* $\mathcal{F}$ *such that* $\lambda(k)^{-1}B_{\mathrm{PT}}(k)$ *converges to* $B_0(0)$ *on* $D(H_0)$ *as* $k \to 0$.

If there exists a ground state Ψ_{QFT} *such that*

$$\frac{1}{\omega(\cdot)}\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\mathrm{QFT}}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^d) \qquad (3.20)$$

for a vector $\Phi \in D(T_{\mathrm{PT}})$, *then* $(\Phi\,,\,B_0(0)\Psi_{\mathrm{QFT}})_{\mathcal{F}} = 0$.

Proof . Suppose there exists a ground state Ψ_{QFT} satisfying the condition (3.20) for a vector $\Phi \in D(T_{\mathrm{PT}})$. Since $\Phi \in D(T_{\mathrm{PT}})$, we have

$$\left(\Phi\,,\,(\widehat{H}_{\mathrm{QFT}}+\omega(\cdot))^{-1}B_{\mathrm{PT}}(\cdot)\Psi_{\mathrm{QFT}}\right)_{\mathcal{F}} \in L^2(\mathbb{R}^d)$$

by the definition (3.14). So, we can define $F \in L^2(\mathbb{R}^d)$ by

$$
\begin{aligned}
F(k) \quad := \quad & \left(\Phi \,,\, (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \\
& + \frac{1}{\omega(k)} \left(\Phi \,,\, (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} \widehat{H}_{\mathrm{QFT}} B_{\mathrm{PT}}(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} ,
\end{aligned}
$$

where we used the condition (3.20) in the second term of RHS. Since $(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)$ is bounded for every $k \in \mathbb{R}^d \setminus \{0\}$ by (Ass.2), we have

$$
\begin{aligned}
& \left[B_{\mathrm{PT}}(k) \,,\, (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} \right] \\
= \quad & (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} \left[\widehat{H}_{\mathrm{QFT}} \,,\, B_{\mathrm{PT}}(k) \right] (\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} . \quad (3.21)
\end{aligned}
$$

So, Eq. (3.21) leads us to:

$$
F(\cdot) = \frac{1}{\omega(\cdot)} \left(\Phi \,,\, B_{\mathrm{PT}}(\cdot) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}}
$$

as an identity on $L^2(\mathbb{R}^d_{>\varepsilon})$ for every $\varepsilon > 0$. Set $B_0(k)$ as $B_0(k) := \lambda(k)^{-1} B_{\mathrm{PT}}(k)$ for $k \in \mathbb{R}^d$ with $0 < |k| < \varepsilon_0$. Then, the above equation implies

$$
\int_{\varepsilon < |k| < \varepsilon_0} \frac{|\lambda(k)|^2}{\omega(k)^2} \left| \left(\Phi \,,\, B_0(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 dk = \int_{\varepsilon < |k| < \varepsilon_0} |F(k)|^2 dk .
$$

Since $F \in L^2(\mathbb{R}^d)$, taking $\varepsilon \to 0$ and using Lebesgue's monotone convergence theorem, we have

$$
\frac{\lambda(\cdot)}{\omega(\cdot)} \left(\Phi \,,\, B_0(\cdot) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \in L^2(\mathbb{R}^d_{<\varepsilon_0}). \quad\quad (3.22)
$$

Since $\left| \left(\Phi \,,\, B_0(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 \to \left| \left(\Phi \,,\, B_0(0) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2$ as $k \to 0$ by the assumption (3), for every $\varepsilon > 0$ there exists a positive number $\delta_\Phi(\varepsilon) > 0$ such that

$$
\left| \left(\Phi \,,\, B_0(0) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 - \varepsilon \leq \left| \left(\Phi \,,\, B_0(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2
$$

for every k with $|k| < \delta_\Phi(\varepsilon)$. Set $\varepsilon_0 \wedge \delta_\Phi(\varepsilon)$ as $\varepsilon_0 \wedge \delta_\Phi(\varepsilon) := \min\{\varepsilon_0, \delta_\Phi(\varepsilon)\}$. Then, we have

$$
\begin{aligned}
& \left\{ \left| \left(\Phi \,,\, B_0(0) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 - \varepsilon \right\} \int_{|k| < \varepsilon_0 \wedge \delta_\Phi(\varepsilon)} \frac{|\lambda(k)|^2}{\omega(k)^2} dk \\
\leq \quad & \int_{|k| < \varepsilon_0} \frac{|\lambda(k)|^2}{\omega(k)^2} \left| \left(\Phi \,,\, B_0(k) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 dk < \infty
\end{aligned}
$$

by the condition (3.22). So, by the assumption (2) we are bound to conclude that $\left| \left(\Phi \,,\, B_0(0) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} \right|^2 \leq \varepsilon$. Thus, taking $\varepsilon \to 0$ yields $\left(\Phi \,,\, B_0(0) \Psi_{\mathrm{QFT}} \right)_{\mathcal{F}} = 0$. $\qquad\qquad \square$

We state a useful domain property which causes the absence of the mass gap. Let $R(\widehat{H}_{\mathrm{QFT}})$ denote the range of $\widehat{H}_{\mathrm{QFT}}$:

$$
R(\widehat{H}_{\mathrm{QFT}}) := \left\{ \widehat{H}_{\mathrm{QFT}} \Psi \,\middle|\, \Psi \in D(\widehat{H}_{\mathrm{QFT}}) \right\} .
$$

Theorem 3.10 *Assume there is a ground state of H_{QFT}. If $B_{\mathrm{PT}}(k)$ is a bounded operator on $\mathcal{F}$ for every $k \in \mathbb{R}^d \setminus \{0\}$ such that $\|B_{\mathrm{PT}}(\cdot)\|_{\mathcal{B}(\mathcal{F})} \in L^2(\mathbb{R}^d)$, then $R(\widehat{H}_{\mathrm{QFT}}) \subset D(T_{\mathrm{PT}})$.*

Proof . For every $\Phi \in R(\widehat{H}_{\mathrm{QFT}})$, there is a $\Psi \in D(\widehat{H}_{\mathrm{QFT}})$ such that $\Phi = \widehat{H}_{\mathrm{QFT}}\Psi$. Thus, for every $\varepsilon > 0$ we have the following estimate:

$$
\begin{aligned}
& \int_{\varepsilon < |k| < \varepsilon^{-1}} | \, (K_{\mathrm{PT}}(k) \, , \, \Phi)_{\mathcal{F}} \, |^2 dk \\
= \;\; & \int_{\varepsilon < |k| < \varepsilon^{-1}} \left| \left(\widehat{H}_{\mathrm{QFT}}(\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_{\mathrm{PT}}(k)\Psi_{\mathrm{QFT}} \, , \, \Psi \right)_{\mathcal{F}} \right|^2 dk \\
\leq \;\; & \|\Psi\|_{\mathcal{F}}^2 \int_{\mathbb{R}^d} \|B_{\mathrm{PT}}(k)\|_{\mathcal{B}(\mathcal{F})}^2 dk < \infty.
\end{aligned}
$$

Therefore, taking $\varepsilon \to 0$, together with Lebesgue's monotone convergence theorem, yields $\Phi \in D(T_{\mathrm{PT}})$. $\square$

4 IR Catastrophe and Absence of Grand State

In the case where IR singularity of T_{PT} is determined by the singularity of the function λ on $\mathbb{R}^d$ as seen in Theorem 3.9, we introduce a notion for the order of IR singularity of the maximal Carleman operator T_{PT} at $k = 0$. That is, in this section we assume that there is a measurable function λ on $\mathbb{R}^d$ satisfying the condition (1) of Theorem 3.9 and an operator $B_0(k)$ acting in $\mathcal{F}$ for every $k \in \mathbb{R}^d$ such that the following (S1) and (S2) are satisfied:

(S1) $B_{\mathrm{PT}}(k) = \lambda(k)B_0(k)$ on $D(H_0)$ for every $k \in \mathbb{R}^d \setminus \{0\}$;

(S2) $B_0(k)\Psi \longrightarrow B_0(0)\Psi$ in $\mathcal{F}$ as $k \to 0$ for every $\Psi \in D(H_0)$.

IR singularity condition [4, 5] is reinterpreted as $\lambda/\omega \notin L^2(\mathbb{R}^d)$ for the Carleman operator T_{PT}. We extend the notion of IR singularity condition. Our new notion is:

Definition 4.1 We say ω and λ satisfy IR singularity condition if there are constants $\gamma_1, \gamma_2, \varepsilon_2 > 0$ such that $\lambda/\omega^\gamma \in L^2(\mathbb{R}^d)$ for every γ with $\gamma < \gamma_1$ and $\lambda/\omega^\gamma \notin L^2(\mathbb{R}^d_{<\varepsilon})$ for every γ and ε with $\gamma > \gamma_2$ and $\varepsilon_2 \geq \varepsilon > 0$. We say γ is in the IR-safe region (resp. the IR-divergent region) if $\gamma < \gamma_1$ (resp. $\gamma > \gamma_2$). In particular, we call γ_{c} the order of IR singularity condition when $\gamma_1 = \gamma_2 = \gamma_{\mathrm{c}}$ and $\lambda/\omega^{\gamma_{\mathrm{c}}} \notin L^2(\mathbb{R}^d_{<\varepsilon})$ for every ε with $\varepsilon_2 \geq \varepsilon > 0$. In this case, we also say $\gamma = \gamma_{\mathrm{c}}$ is in the IR-divergent region.

Example 4.1 *Most standard assumptions for ω and λ are $\lambda/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and $\lambda/\omega \notin L^2(\mathbb{R}^d)$. So, in this case, ω and λ satisfy IR singularity condition and $1/2 < \gamma_{\mathrm{c}} \leq 1$.*

We say *a symmetric operator S strongly commutes with H_{QFT}* if $e^{itH_{\mathrm{QFT}}}S \subset Se^{itH_{\mathrm{QFT}}}$ for all $t \in \mathbb{R}$. Then, we can derive the following theorem from Theorem 3.7. This is a generalization of Dereziński and Gérard's [13, Lemma 2.6] and ours [5, Theorem 3.4].

Theorem 4.2 *Suppose $D(H_{\mathrm{QFT}}) = D(H_0)$. Assume ω and λ satisfy IR singularity condition with the order γ_{c} less than or equal to 1 (i.e., $\gamma_{\mathrm{c}} \le 1$). Then, there is no ground state Ψ_{QFT} satisfying all of the following (i)–(iii):*

(i) $B_0(0)$ *is symmetric and strongly commutes with H_{QFT}.*

(ii) $B_0(0)\Psi_{\mathrm{QFT}} \ne 0$.

(iii) $\sup_{k \in \mathbb{R}^d} \omega(k)^{\gamma-1} \|(B_0(k) - B_0(0))\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} < \infty$ *for some γ in the IR-safe region.*

Proof . We use the reduction of absurdity. So, we suppose that a ground state Ψ_{QFT} exists such that all of (i)–(iii) hold. For all $\Phi \in D(N^{1/2})$ and every $k \in \mathbb{R}^d \setminus \{0\}$ we have

$$
\begin{aligned}
(K_{\mathrm{PT}}(k)\,,\,\Phi)_{\mathcal{F}} &= \lambda(k)\left((\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1}B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi\right)_{\mathcal{F}} \\
&\quad + \lambda(k)\left((\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1}(B_0(k) - B_0(0))\Psi_{\mathrm{QFT}}\,,\,\Phi\right)_{\mathcal{F}} \\
&= \frac{\lambda(k)}{\omega(k)}(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}} \\
&\quad + \lambda(k)\left((\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1}(B_0(k) - B_0(0))\Psi_{\mathrm{QFT}}\,,\,\Phi\right)_{\mathcal{F}}
\end{aligned}
$$

by (i). This equation implies

$$
\begin{aligned}
0 &\le |(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}}|^2 \int_{|k|>\varepsilon} \frac{|\lambda(k)|^2}{\omega(k)^2}dk \\
&\le 2\int_{\mathbb{R}^d}|(K_{\mathrm{PT}}(k)\,,\,\Phi)_{\mathcal{F}}|^2 dk \\
&\quad + 2\|\Phi\|_{\mathcal{F}}^2\left(\sup_{k \in \mathbb{R}^d}\omega(k)^{\gamma-1}\|(B_0(k) - B_0(0))\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}\right)^2 \int_{\mathbb{R}^d}\frac{|\lambda(k)|^2}{\omega(k)^{2\gamma}}dk.
\end{aligned}
$$

Here we note that the two integrals of RHS are finite by Theorem 3.7 and (iii), and they are independent of $\varepsilon > 0$. Taking $\varepsilon \to 0$, Lebesgue's monotone convergence theorem tells us that $(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)$ is bound to be 0 (i.e., $(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi) = 0$) for all $\Phi \in D(N^{1/2})$ since $\lambda/\omega \notin L^2(\mathbb{R}^d)$. Since $D(N^{1/2})$ is dense in $\mathcal{F}$, we reach $B_0(0)\Psi_{\mathrm{QFT}} = 0$ finally, which contradicts (ii). $\qquad\square$

We can obtain Dereziński and Gérard's [13, Lemma 2.6] as a corollary of Theorem 4.2:

Corollary 4.3 *Suppose $D(H_{\mathrm{QFT}}) = D(H_0)$ and $B_{\mathrm{PT}}(k)$ can be decomposed into $B_{\mathrm{PT}}(k) = g(k)I \otimes I + J_{\mathrm{err}}(k)$ on $D(H_0)$ for every $k \in \mathbb{R}^d \setminus \{0\}$. Assume the following (1)–(3):*

(1) $g/\omega \notin L^2(\mathbb{R}^d)$,

(2) $g/\omega^{\gamma_0} \in L^2(\mathbb{R}^d)$ *for a* γ_0 *with* $0 < \gamma_0 < 1$,

(3) $g(k)^{-1}J_{\mathrm{err}}(k)\Psi \longrightarrow 0$ *as* $k \to 0$ *for every* $\Psi \in D(H_0)$.

Then, there is no ground state Ψ_{QFT} *satisfying*

$$\sup_{k\in\mathbb{R}^d} \omega(k)^{\gamma_0-1}g(k)^{-1}\|J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} < \infty. \qquad (4.1)$$

Proof. Set $B_0(0)$, $\lambda(k)$, and $B_0(k)$ as $B_0(0) := I \otimes I$, $\lambda(k) := g(k)$, and $B_0(k) := \lambda(k)^{-1}J_{\mathrm{err}}(k) + I \otimes I$, respectively. Then, the assumption (3) implies (S1), (S2), and (i) and (ii) of Theorem 4.2. The assumptions (1) and (2) tell us that ω and λ satisfy IR singularity condition such that 1 is in the IR-divergent region and γ_0 in the IR-safe region. Thus, Theorem 4.2 concludes that there is no ground sate satisfying

$$\sup_{k\in\mathbb{R}^d} \omega(k)^{\gamma_0-1}g(k)^{-1}\|J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}}$$
$$= \sup_{k\in\mathbb{R}^d} \omega(k)^{\gamma_0-1}\|(B_0(k) - B_0)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} < \infty.$$

$\square$

As a corollary of Theorem 4.2 we also obtain [5, Theorem 3.4] of which statement can be applied to several models as well as in [5]:

Corollary 4.4 *Suppose* $D(H_{\mathrm{QFT}}) = D(H_0)$ *and there is an operator* C_{PT} *with* $D(C_{\mathrm{PT}}) \supset D(H_0)$ *such that* $B_{\mathrm{PT}}(k) = \lambda(k)C_{\mathrm{PT}}$ *on* $D(H_0)$ *for all* $k \in \mathbb{R}^d$. *Assume the following (1)–(3):*

(1) $\lambda/\omega \notin L^2(\mathbb{R}^d)$,

(2) $\lambda/\omega^{\gamma_0} \in L^2(\mathbb{R}^d)$ *for a* γ_0 *with* $0 < \gamma_0 < 1$,

(3) C_{PT} *is symmetric and strongly commutes with* H_{QFT}.

Then, there is no ground state Ψ_{QFT} *satisfying* $C_{\mathrm{PT}}\Psi_{\mathrm{QFT}} \neq 0$.

Proof. Set $B_0(k)$ as $B_0(k) := C_{\mathrm{PT}}$ for all $k \in \mathbb{R}^d$. Then, (S1), (S2), and (i) of Theorem 4.2 hold by the assumption (3). The assumption (1) implies $\gamma_c \leq 1$. Since $B_0(k) - B_0(0) = 0$ on $D(H_0)$ now, the condition (iii) in Theorem 4.2 always holds for γ_0 in the IR-safe region by the assumption (2). Thus, Theorem 4.2 leads us to the conclusion that there is no ground state Ψ_{QFT} satisfying $C_{\mathrm{PT}}\Psi_{\mathrm{QFT}} = B_0(0)\Psi_{\mathrm{QFT}} \neq 0$. $\square$

The following theorem follows from Theorem 3.9:

Theorem 4.5 *Assume* $D(H_{\mathrm{QFT}}) = D(H_0)$ *and* $\lambda/\omega \notin L^2(K)$ *for every neighborhood* K *of* $k = 0$. *Then, there is no ground state* Ψ_{QFT} *in* $D(T_{\mathrm{PT}})$ *satisfying* $\langle B_0(0)\rangle_{\mathrm{gs}} \neq 0$. *Thus, in particular, if* $B_0(0)\Psi \neq 0$ *for every* $\Psi \in D(H_0)$, *then no ground state exists in* $D(T_{\mathrm{PT}})$, *namely, IR catastrophe occurs.*

Proof . Let us suppose there is a ground state Ψ_{QFT} in $D(T_{\text{PT}})$ now. We easily have

$$\frac{1}{\omega(\cdot)}\left(\Psi_{\text{QFT}}\,,\,(\widehat{H}_{\text{QFT}}+\omega(\cdot))^{-1}\widehat{H}_{\text{QFT}}B_{\text{PT}}(\cdot)\Psi_{\text{QFT}}\right)_{\mathcal{F}}=0.$$

Thus, it follows immediately from Theorem 3.9 that $\langle B_0(0)\rangle_{\text{gs}} = (\Psi_{\text{QFT}}\,,$ $B_0(0)\Psi_{\text{QFT}}\,)_{\mathcal{F}} = 0$, which means our theorem holds. $\qquad\square$

Remark 4.1 *As mentioned in Section 1 (i.e., [25, p.212 and p.213]), since the full model of non-relativistic quantum electrodynamics [9, 22] has local gauge invariance, the commutation relation $i[H_{\text{QFT}}\,,\,x] = v$ holds for the full model, where x and v are the position and velocity of the non-relativistic electron. This commutation relation cancels IR singularity in Eqs.(1.1) and (3.2). Thus, IR catastrophe does not occur for the full model. Since $B_0(0) = v$ and the above commutation relation implies $\langle v\rangle_{\text{gs}} = 0$ for the full model, we have $\langle B_0(0)\rangle_{\text{gs}} = 0$. Namely, local gauge invariance also works to avoid IR catastrophe in Theorem 4.5.*

Remark 4.2 *Theorem 4.5 is a general expression of IR catastrophe for the Nelson Hamiltonian [10, 27] and for the spin-boson Hamiltonian [5, 6, 30].*

Let us denote by $\sigma_{\text{ess}}(S)$ the essential spectrum of a self-adjoint operator S.

Combining Lemma 3.10 and Theorem 4.5 yields the following theorem, which states IR singularity condition prohibits H_{QFT} from making the mass gap:

Theorem 4.6 *Suppose $D(H_{\text{QFT}}) = D(H_0)$. Assume the following (1)–(3):*

(1) $\lambda/\omega \notin L^2(\mathbb{R}^d)$,

(2) *$B_{\text{PT}}(k)$ is a bonded operator acting on $\mathcal{F}$ for every $k \in \mathbb{R}^d \setminus \{0\}$ such that $\|B_{\text{PT}}(\cdot)\|_{\mathcal{B}(\mathcal{F})} \in L^2(\mathbb{R}^d)$,*

(3) $B_0(0)\Psi \neq 0$ *for every $\Psi \in D(H_0)$.*

Then, there is no gap between the ground state energy and the infimum of the essential spectrum of H_{QFT}:

$$E_0(H_{\text{QFT}}) = \inf \sigma_{\text{ess}}(H_{\text{QFT}}).$$

Proof . We prove our statement by the reduction of absurdity. So, suppose $E_0(H_{\text{QFT}}) < \inf \sigma_{\text{ess}}(H_{\text{QFT}})$. Then, there is a ground state Ψ_{QFT} and $0 < \inf \sigma_{\text{ess}}(\widehat{H}_{\text{QFT}})$. Thus, for every $\Phi \in D(\widehat{H}_{\text{QFT}}) \cap \ker(\widehat{H}_{\text{QFT}})^{\perp}$, we have $\inf \sigma_{\text{ess}}(\widehat{H}_{\text{QFT}})\|\Phi\|_{\mathcal{F}} \leq \|\widehat{H}_{\text{QFT}}\Phi\|_{\mathcal{F}}$. This inequality is equivalent to the fact that $R(\widehat{H}_{\text{QFT}})$ is closed as well known. So, we have $\mathcal{F} = \ker(\widehat{H}_{\text{QFT}}) \bigoplus R(\widehat{H}_{\text{QFT}})$. Let P_0 be the orthogonal projection onto $\ker(\widehat{H}_{\text{QFT}})$. For every $\Psi \in \mathcal{F}$ with $P_0\Psi \neq 0$, there are $\Psi_n \in D(T_{\text{PT}})$, $n \in \mathbb{N}$, such that $\Psi_n \to \Psi$ as $n \to \infty$, since $D(T_{\text{PT}})$ is dense in $\mathcal{F}$ by Theorem 3.7. We have $P_0\Psi_n \neq 0$ for almost all $n \in \mathbb{N}$ except finite n's because $P_0\Psi \neq 0$. We note $\Psi_n = P_0\Psi_n + (I - P_0)\Psi_n$

and $(I - P_0)\Psi_n \in R(\widehat{H}_{\mathrm{QFT}}) \subset D(T_{\mathrm{PT}})$ by Lemma 3.10. Thus, we obtain $0 \neq P_0\Psi_n = \Psi_n - (I - P_0)\Psi_n \in D(T_{\mathrm{PT}})$. On the other hand, $P_0\Psi_n \notin D(T_{\mathrm{PT}})$ by Theorem 4.5 since $P_0\Psi_n \neq 0$ is also a ground state of $\widehat{H}_{\mathrm{QFT}}$. Therefore, we reach a contradiction.

$\square$

As explained in Example 5.1 below, we need another statement to avoid the restriction coming from (ii) in Theorem 4.2. We take account of the order of IR singularity condition. Then, we obtain the following from Theorem 3.7:

Theorem 4.7 *Suppose $D(H_{\mathrm{QFT}}) = D(H_0)$ and $B_0(0)$ is symmetric and strongly commutes with H_{QFT}. Assume there is an $\varepsilon_0 > 0$ and operators $B_j(k)$, $j = 1, \cdots, d$, acting in $\mathcal{F}$ for every $k \in \mathbb{R}^d \setminus \{0\}$ such that $B_0(k)\Psi_{\mathrm{QFT}}$ is decomposed into $B_0(k)\Psi_{\mathrm{QFT}} = B_0(0)\Psi_{\mathrm{QFT}} + \sum_{j=1}^d k_j B_j(k)\Psi_{\mathrm{QFT}}$ for $|k| < \varepsilon_0$. If ω and λ satisfy IR singularity condition with the order γ_{c}, and moreover,*

$$\int_{|k|<\varepsilon_0} \frac{|k_j||\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk < \infty$$

for a $\gamma > 0$ with $\gamma < \gamma_{\mathrm{c}} < (1+\gamma)/2$ and $j = 1, \cdots, d$, then there is no ground state Ψ_{QFT} satisfying $B_0(0)\Psi_{\mathrm{QFT}} \neq 0$ and $\sup_{|k|<\varepsilon_0} \|B_j(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} < \infty$ for all $j = 1, \cdots, d$.

Proof. We use the reduction of absurdity. So, we suppose that there is such a ground state Ψ_{QFT}. Let us fix $\Phi \in D(N^{1/2})$ arbitrarily and define a function $F_\Phi(k)$ by $F_\Phi(k) := (K_{\mathrm{PT}}(k), \Phi)_{\mathcal{F}}$. We define another function $F_\gamma(k)$ by $F_\gamma(k) := \lambda(k)\omega(k)^{-\gamma}$. Then, we have $F_\Phi \in L^2(\mathbb{R}^d)$ by Theorem 3.7 and $F_\gamma \in L^2(\mathbb{R}^d)$ by our assumption. For every ε with $\varepsilon < \min\{\varepsilon_0, \varepsilon_2\} =: \varepsilon_0 \wedge \varepsilon_2$, where ε_2 is in Definition 4.1, we have

$$
\begin{aligned}
&\int_{\varepsilon<|k|<\varepsilon_0\wedge\varepsilon_2} F_\Phi(k)F_\gamma(k)dk \\
=\ & (B_0(0)\Psi_{\mathrm{QFT}}, \Phi)_{\mathcal{F}} \int_{\varepsilon<|k|<\varepsilon_0\wedge\varepsilon_2} \frac{|\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk \\
&+ \sum_{j=1}^d \int_{\varepsilon<|k|<\varepsilon_0\wedge\varepsilon_2} \frac{k_j|\lambda(k)|^2}{\omega(k)^\gamma} \\
&\qquad \left((\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_j(k)\Psi_{\mathrm{QFT}}, \Phi\right)_{\mathcal{F}} dk. \qquad (4.2)
\end{aligned}
$$

In the first term of RHS of the above, we used the assumption that $B_0(0)$ commutes with H_{QFT}. We can estimate the last integrals as:

$$
\begin{aligned}
&\left| \int_{\varepsilon<|k|<\varepsilon_0\wedge\varepsilon_2} \frac{k_j|\lambda(k)|^2}{\omega(k)^\gamma} \left((\widehat{H}_{\mathrm{QFT}} + \omega(k))^{-1} B_j(k)\Psi_{\mathrm{QFT}}, \Phi\right)_{\mathcal{F}} dk \right| \\
\leq\ & \|\Phi\|_{\mathcal{F}} \sup_{|k|<\varepsilon_0} \|B_j(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} \int_{|k|<\varepsilon_0} \frac{|k_j||\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk < \infty. \qquad (4.3)
\end{aligned}
$$

Combining Eq.(4.2) and the inequality (4.3) gives us the inequality:

$$
\begin{aligned}
0 \quad &\leq \quad \left| (B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}} \right| \int_{\varepsilon < |k| < \min\{\varepsilon_0, \varepsilon_2\}} \frac{|\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk \\
&\leq \quad \|F_\Phi\|_{L^2(\mathbb{R}^d)} \|F_\gamma\|_{L^2(\mathbb{R}^d)} \\
&\quad + \|\Phi\|_{\mathcal{F}} \sum_{j=1}^{d} \sup_{|k|<\varepsilon_0} \|B_j(k)\Psi_{\mathrm{QFT}}\|_{\mathcal{F}} \int_{|k|<\varepsilon_0} \frac{|k_j||\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk < \infty.
\end{aligned}
$$

Taking $\varepsilon \to 0$, Lebesgue's monotone convergence theorem tells us that $(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}}$ is bound to be 0 (i.e., $(B_0(0)\Psi_{\mathrm{QFT}}\,,\,\Phi)_{\mathcal{F}} = 0$) for all $\Phi \in D(N^{1/2})$ since $\lambda/\omega^{(1+\gamma)/2} \notin L^2(\mathbb{R}^d)$. Since $D(N^{1/2})$ is dense in $\mathcal{F}$, we reach $B_0(0)\Psi_{\mathrm{QFT}} = 0$ finally. This is a contradiction. $\qquad\square$

5 An Application

In this section, we consider the model of a non-relativistic electron coupled with a Bose field made from several sorts of phonons [37, Chap.4] or polaritons [35, §11.4] in a material such as a crystal or a metal. Then, the order of IR singularity condition depends on the sorts of phonons or polaritons. Because each dispersion relation $\omega(k)$ is determined by an individual dispersion equation derived from the equation of motion of atoms in the material (see [35, 37] for theoretical understanding and [1, 18, 34, 50, 52, 57] for experimental understanding). In addition, of course, the interaction function $\rho(k)$ depends on the property of the material. As in Eq.(5.1) of Example 5.1, we idealize $\omega(k)$ and $\rho(k)$ in Theorem 5.5 mathematically to investigate the order of IR singularity.

We put the non-relativistic electron in the material. We suppose that the electron is negatively charged and thus is attracted by a plus-charged source which is caused by the positively charged ion cores caused by, for instance, the crystal lattice deformation [35, §10.3] (also called the crystal lattice distortion [14, 15]). Thus, as the operator A in Eq.(2.4) we employ a Hamiltonian H_{at} given by the Schrödinger operator with a potential V:

$$
H_{\mathrm{at}} \equiv \frac{1}{2}p^2 + V
$$

acting in $\mathcal{H} = L^2(\mathbb{R}^d)$, where $p := -i\nabla_x$ is the momentum of the electron. We use the natural units here.

As in [27] we consider potentials V in the class either (N1-1) or (N1-2) below. Here we say that *V is in class (N1-1)* (resp. *(N1-2)*) if the following (N1-1-1) and (N1-1-2) (resp. (N1-2-1) and (N1-2-2)) hold. These conditions are set so that if V is in class (N1-1) or (N1-2), then H_{at} becomes a self-adjoint operator bounded from below with $D(H_{\mathrm{at}}) \subset D(p^2)$, and moreover, H_{at} has a ground state ψ_{at}. When we say that we assume (N1), we mean that either (N1-1) or (N1-2) is assumed.

(N1-1) [2]:

(N1-1-1) H_{at} is self-adjoint on $D(H_{at}) \equiv D(p^2) \cap D(V)$ and bounded from below,

(N1-1-2) there exist positive constants c_1 and c_2 such that $|x|^2 \leq c_1 V(x) + c_2$ for almost every (a.e.) $x \in \mathbb{R}^d$, and $\int_{|x| \leq R} |V(x)|^2 dx < \infty$ for all $R > 0$.

(N1-2) [55]:

(N1-2-1) $V \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and $\lim_{|x| \to \infty} |V(x)| = 0$.

Following [48, Theorem X15] and [49, §XIII.4, Example 6] the condition (N1-2-1) implies that H_{at} is self-adjoint on $D(p^2)$; V is infinitesimally p^2-bounded; and the essential spectrum $\sigma_{ess}(H_{at})$ of H_{at} is equal to $[0, \infty)$. So, we assume the following in addition:

(N1-2-2) H_{at} has a ground state ψ_{at} satisfying $\psi_{at}(x) > 0$ for a.e. $x \in \mathbb{R}^d$ and $E_{at} := inf\sigma(H_{at}) < 0$.

In order to define the interaction Hamiltonian H_I of the models, we use the fact that $\mathcal{F}$ is unitarily equivalent to the constant fiber direct integral $L^2(\mathbb{R}^d; \mathcal{F}_b)$, i.e.,

$$\mathcal{F} \equiv L^2(\mathbb{R}^d) \otimes \mathcal{F}_b \cong L^2(\mathbb{R}^d; \mathcal{F}_b) \equiv \int_{\mathbb{R}^d}^\oplus \mathcal{F}_b dx$$

(see [49, 53]). Throughout this section, we identify $\mathcal{F}$ to the constant fiber direct integral, i.e.,

$$\mathcal{F} = \int_{\mathbb{R}^d}^\oplus \mathcal{F}_b dx.$$

If a measurable function $\rho(k)$ satisfy $1^{<\Lambda}\rho \in L^2(\mathbb{R}^d)$, we give the interaction Hamiltonian H_I by the so-called Fröhlich interaction [16]:

$$H_I := q \int_{\mathbb{R}^d}^\oplus \{a(1^{<\Lambda}\rho e^{-ikx}) + a^\dagger(1^{<\Lambda}\rho e^{-ikx})\} dx$$

for every $q \in \mathbb{R}$. Symbolically using the kernels of the annihilation and creation operators, the interaction Hamiltonian H_I is often expressed as

$$H_I = q \int_{|k|<\Lambda} \left(\rho(k) e^{ikx} a(k) + \overline{\rho(k)} e^{-ikx} a^\dagger(k) \right) dk.$$

We also assume the following:

(N2) $1^{<\Lambda}\rho$, $1^{<\Lambda}\rho/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

The Hamiltonian H_{QFT} of the models we consider in this section is given by

$$
\begin{aligned}
H_{QFT} &:= H_0 + H_I \\
&= H_{at} \otimes I + I \otimes H_b + q \int_{\mathbb{R}^d}^\oplus \{a(1^{<\Lambda}\rho e^{-ikx}) + a^\dagger(1^{<\Lambda}\rho e^{-ikx})\} dx
\end{aligned}
$$

acting in $\mathcal{F}$ [38, 39]. Then, we call this H_{QFT} the *Lee-Low-Pines (LLP) Hamiltonian* in this paper, though it is called the Pauli-Fierz Hamiltonian in [13, 19, 20]. Because we are interested in the models in solid state physics.

As explained in [27], we have the following assertion:

Proposition 5.1 *Assume (N1) and (N2). Then, H_{QFT} is self-adjoint with $D(H_{\mathrm{QFT}}) = D(H_0) \equiv D(H_{\mathrm{at}} \otimes I) \cap D(I \otimes d\Gamma(1))$. H is bounded from below for arbitrary values of q.*

Once we assume the existence of a ground state, it has to have the property of the spatial localization as stated in Propositions 5.2 and 5.3 below.

In the same way as in [27, Proposition 6.1] we can prove the following:

Proposition 5.2 *Assume (N1-1) and (N2). If H_{QFT} has a ground state Ψ_{QFT}, then $\Psi_{\mathrm{QFT}} \in D(x^2 \otimes I)$.*

In the same way as in [27, Proposition 6.3], obeying the idea in [22] with a little modification to meet our models, we have the following:

Proposition 5.3 *Assume (N1-2) and (N2). If H_{QFT} has a ground state Ψ_{QFT}, then there is $C_0 > 0$ such that $\Psi_{\mathrm{QFT}} \in D(e^{C_0|x|})$.*

Remark 5.1 *Propositions 5.2 and 5.3 tell us that if LLP Hamiltonian has a ground state, the uncertainty $\Delta_{\mathrm{gs}}|x| := \langle(|x| - \langle|x|\rangle_{\mathrm{gs}})^2\rangle_{\mathrm{gs}}^{1/2}$ of $|x|$ in the ground state is finite. This is a natural fact in quantum theory to observe the electron in the ground state. On the other hand, Theorem 4.5 tells us that IR singularity condition causes IR catastrophe for LLP Hamiltonian. Thus, since the electron has to dress itself in the cloud of infinitely many soft bosons, we can expect that [27, Theorem 2.1] also holds for LLP Hamiltonian. Namely, $\Delta_{\mathrm{gs}}|x|$ must diverge. Therefore, supposing the existence of a ground state under IR singularity condition brings about a contradiction in quantum theory. We find this contradiction in a logic of mathematics to show Theorem 5.5 below.*

The following OPPT formula can be proved in the same way as in [27, Proposition 3.1]:

Proposition 5.4 *Assume (N1) and (N2). Then, for all $f \in C_0^\infty \ (\mathbb{R}^d \setminus \{0\})$,*

$$a(f)\Psi_{\mathrm{QFT}} = -q \int_{\mathbb{R}^d} \overline{f(k)}\rho(k) \left(\widehat{H}_{\mathrm{QFT}} + \omega(k)\right)^{-1} e^{-ikx}\Psi_{\mathrm{QFT}}dk,$$

provided that $\Psi_{\mathrm{QFT}} \in D(x^2 \otimes I)$. Therefore,

$$B_{\mathrm{PT}}(k) = q1^{<\Lambda}(k)\rho(k)e^{-ikx} \otimes I.$$

To consider the problem mentioned in Section 1 (i.e., the problem stated in [27, Remark 2]), we give an example of $\omega(k)$ and $\rho(k)$ here:

Example 5.1 *As an example of the dispersion relation $\omega(k)$ and the interaction function $\rho(k)$, through a simplification and an idealization, let us set them as*

$$\omega(k) = |k|^\mu \ \ and \ \ \rho(k) = |k|^{-\nu} \tag{5.1}$$

for $\mu \geq 0$ and $\nu \in \mathbb{R}$, respectively. Because we are interested in IR situation around $k = 0$. Then, we have

$$\gamma_{\mathrm{c}} = \frac{d - 2\nu}{2\mu}. \tag{5.2}$$

Here we note we can consider γ_{c} to be infinite when $\mu = 0$ because (N2) requires that ν should be less than $d/2$ (i.e., $\nu < d/2$) and thus all γ are in the IR-safe region in this case. The condition, $d \leq 2(\mu + \nu)$, implies $\lambda/\omega \notin L^2(\mathbb{R}^d)$. A sufficient condition so that we can obtain γ_0 in (2) of Corollary 4.3 and Eq.(4.1) is $d > 2(\mu + \nu - 1)$ as shown in the proof of (iii) of Theorem 5.5. So, because H_{QFT} should be defined to be self-adjoint, a sufficient condition so that Corollary 4.3 works is

$$\max\left\{\frac{\mu}{2} + \nu,\, \mu + \nu - 1\right\} < \frac{d}{2} \leq \mu + \nu. \tag{5.3}$$

As in Example 5.1 the dimension d has haven a restriction from below if we use Corollary 4.3. However, since $(\mu/2) + \nu < \mu + \nu - 1$ iff $\mu > 2$, there is a possibility that $(\mu/2) + \nu < d/2 \leq \mu + \nu - 1$ when $\mu > 2$. Thus, Corollary 4.3 does not work in this case. We try to remove this restriction in the case $\mu > 2$ by using Theorem 4.7 from now on.

Let us take μ and γ with $2 < \mu$ and $0 < \gamma < 1 - (2/\mu)$ now. If ν satisfies

$$\frac{d}{2} - \frac{1 + \gamma}{2}\mu \leq \nu < \min\left\{\frac{d + 1}{2} - \frac{1 + \gamma}{2}\mu,\, \frac{d}{2}\right\}, \tag{5.4}$$

then we have

$$\gamma\mu + \nu < \frac{1 + \gamma}{2}\mu + \nu - \frac{1}{2} < \frac{d}{2} \leq \frac{1 + \gamma}{2}\mu + \nu < \mu + \nu - 1.$$

Namely, d, μ and ν are out of the region (5.3) under Eq.(5.4). But we have the following criterion:

Theorem 5.5 *(Criterion for IR Catastrophe) Set $\omega(k)$ and $\rho(k)$ as Eq.(5.1). Assume that $\mu + 2\nu < d$. Let V is in class (N1) and (N2). Then, the following (i) – (iv) hold:*

(i) If $\nu + \mu < d/2$, then IR catastrophe does not occur, and moreover, there is a constant $q_0 \in \mathbb{R} \cup \{\infty\}$ such that ground state exists in $\mathcal{F}$ for every q with $|q| < q_0$.

(ii) If $\nu + \mu \geq d/2$, then IR catastrophe occurs.

(iii) If d, μ, ν satisfy Eq.(5.3), then there is no ground state in $\mathcal{F}$.

(iv) Set $\gamma := 2\gamma_{\mathrm{c}} - 1$, where γ_{c} is in Eq.(5.2). If $\mu > 2$ and $(d/2) - \mu < \nu < d/2$, then Eq.(5.4) holds and there is no ground state in $\mathcal{F}$.

Proof. We note the condition, $\mu + \nu < d/2$, implies $\lambda/\omega \in L^2(\mathbb{R}^d)$. Hence $\|K_{\mathrm{PT}}(\cdot)\|_{\mathcal{F}} \in L^2(\mathbb{R}^d)$ follows from this condition. Thus, Theorem 3.6 tells us that $\Psi_{\mathrm{QFT}} \in D(N^{1/2})$ if Ψ_{QFT} exists. Namely, IR catastrophe does not

occur. The existence of a ground state Ψ_{QFT} is due to Spohn's result [55]. Thus, part (i) is completed.

Part (ii) follows from Theorem 4.5.

To prove part (iii) we use the reduction of absurdity. Suppose that there is a ground state Ψ_{QFT}. The inequality $d/2 < \mu + \nu$ in Eq.(5.3) implies that $\gamma_{\mathrm{c}} \equiv (d - 2\nu)/2\mu < 1$, so 1 is in the IR-divergent region. Moreover, we have $1 - \mu^{-1} < \gamma_{\mathrm{c}}$ by $\mu + \nu - 1 < d/2$ in Eq.(5.3). Thus, every γ_0 with $1 - \mu^{-1} \le \gamma_0 < \gamma_{\mathrm{c}}$ is in the IR-safe region. Thus, taking $g(k) = q\lambda(k) = q1^{<\Lambda}(k)\rho(k)$, the assumptions (1) and (2) of Corollary 4.3 hold. Taking $J_{\mathrm{err}}(k) = q\lambda(k)(e^{-ikx} - 1) \otimes I$, the assumption (3) of Corollary 4.3 holds. Moreover, since $1 - \mu^{-1} \le \gamma_0$ implies $(\gamma_0 - 1)\mu + 1 \ge 0$, we have

$$\sup_{k \in \mathbb{R}^d} \omega(k)^{\gamma_0 - 1} g(k)^{-1} \| J_{\mathrm{err}}(k)\Psi_{\mathrm{QFT}} \|_{\mathcal{F}}$$

$$= \sup_{k \in \mathbb{R}^d} \omega(k)^{\gamma_0 - 1} \| \left((e^{-ikx} - 1) \otimes I \right) \Psi_{\mathrm{QFT}} \|_{\mathcal{F}}$$

$$\le \left(\sup_{k \in \mathbb{R}^d} |k|^{(\gamma_0 - 1)\mu + 1} \right) \| \left(|x| \otimes I \right) \Psi_{\mathrm{QFT}} \|_{\mathcal{F}}$$

$$\le \| \left(|x| \otimes I \right) \Psi_{\mathrm{QFT}} \|_{\mathcal{F}} < \infty$$

by Propositions 5.2, 5.3 and 5.4. This contradicts the assertion of Corollary 4.3.

Using the reduction of absurdity, we prove part (iv). Thus, we suppose there is a ground state Ψ_{QFT}. Our assumption of (iv) yields Eq.(5.4) immediately. It is clear that γ is in the IR-safe region and $(1 + \gamma)/2$ in the IR-divergent region. Set $\lambda(k)$ and $B_0(k)$ as $\lambda(k) = 1^{<\Lambda}(k)\rho(k)$ and $B_0(k) = qe^{-ikx} \otimes I$ respectively. Then, $B_{\mathrm{PT}}(k) = \lambda(k)B_0(k)$ by Propositions 5.2, 5.3, 5.4. It is easy to check

$$\int_{\mathbb{R}^d} \frac{|k_j||\lambda(k)|^2}{\omega(k)^{1+\gamma}} dk = \int_{\mathbb{R}^d} \frac{|k_j||\lambda(k)|^2}{\omega(k)^{2\gamma_{\mathrm{c}}}} dk \le \int_{\mathbb{R}^d} \frac{|\lambda(k)|^2}{\omega(k)^{2(\gamma_{\mathrm{c}} - (1/2\mu))}} dk < \infty$$

since $\gamma_{\mathrm{c}} - (1/2\mu)$ is in the IR-safe region. We note $B_0(0)\Psi_{\mathrm{QFT}} = q\Psi_{\mathrm{QFT}} \ne 0$. Applying Maclaurin's theorem to $f(t) := e^{-itkx}$ $(t \in [0, 1])$, there is a θ with $0 < \theta < 1$ such that $B_j(k) = -ix_j e^{-i\theta kx} \otimes I$. Thus, Propositions 5.2 and 5.3 lead us to the conclusion that $\sup_{k \in \mathbb{R}^d} \| B_j(k)\Psi_{\mathrm{QFT}} \|_{\mathcal{F}} \le \| |x|\Psi_{\mathrm{QFT}} \|_{\mathcal{F}} < \infty$. However, the last two facts contradict the statement of Theorem 4.7. $\square$

Acknowledgement

References

[1] T. Aizawa, W. Hayami, and S. Otani, Surface phonon dispersion of $ZrB_2(0001)$ and $NbB_2(0001)$, *Phys. Rev. B* **65**, (2001) 024303.

[2] A. Arai, Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, *Rev. Math. Phys.* **13**, (2001) 1075–1094.

[3] A. Arai and M. Hirokawa, On the existence and uniqueness of ground state of a generalized spin-boson model, *J. Funct. Anal.* **151**, (1997) 455–503.

[4] A. Arai and M. Hirokawa, Ground states of a general class of quantum field Hamiltonians, *Rev. Math. Phys.* **12**, (2000) 1085–1135.

[5] A. Arai, M. Hirokawa, and F. Hiroshima, On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoffs, *J. Funct. Anal.* **168**, (1999) 470–497.

[6] A. Arai, M. Hirokawa, and F. Hiroshima, Regularities of ground states of quantum field models, *Kyushu J. Math.* **61**, (2007) 1–52.

[7] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, The renormalized electron mass in non-relativistic quantum electrodynamics, *J. Funct. Anal.* **243**, (2007) 426–535.

[8] V. Bach, J. Fröhlich, and A. Pizzo, Infrared-finite algorithms in QED: The groundstate of an atom interacting with the quantized radiation field, *Commun. Math. Phys.* **264**, (2006) 145–165.

[9] V. Bach, J. Fröhlich, and I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207**, (1999) 249–290.

[10] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos and H. Spohn, Gibbs measure associated with particle-field system, *Rev. Math. Phys.* **14**, (2002) 173–198.

[11] L. Bruneau, private communication.

[12] L. Bruneau and J. Dereziński, Pauli-Fierz Hamiltonians defined as quadratic form, *Rep. Math. Phys.* **54**, (2004) 169–199.

[13] J. Dereziński and C. Gérard, Scattering theory of infrared divergent Pauli-Fierz Hamiltonians, *Ann. H. Poincaré* **5**, (2004) 523–577.

[14] T. Egami, W. Dmowski, R. J. McQueeney, T. R. Sendyka, S. Ishihara, M. Tachiki, H. Yamauchi, S. Tanaka, T. Hinatsu, and S. Uchida, Experimental evidence of local lattice distortion in superconducting oxides, in *Polarons and Bipolarons in High-T_c Superconductors and Related Materials*, Eds. E. K. H. Salje, A. S. Alexandrov, and W. Y. Liang, Cambridge University Press, 1995.

[15] R. P. Feynman, Slow electrons in a polar crystal, *Phys. Rev.* **97**, (1955) 660–665.

[16] H. Fröhlich, Electrons in lattice fields, *Adv. in Phys.* **3**, (1954) 325–362.

[17] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. H. Poincaré* **19**, (1973) 1–103.

[18] L. Fousadier, M. D. Fontana, and W. Kress, Phonon dispersion curves in dilute KTN crystals, *J. Phys.: Condensed Matter* **8**, (1996) 1135–1150.

[19] V. Georgescu, C. Gérard, and J. S. Møller, Spectral theory of massless Pauli-Fierz models, *Commun. Math. Phys.* **249**, (2004) 29–78.

[20] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. H. Poincaré* **1**, (2000) 443–459.

[21] C. Gérard, private communication on interpretation of [13, Eq.(2.9) of Lemma 2.6] in January 2004.

[22] M. Griesemer, E. H. Lieb, and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* **145**, (2001) 557–595.

[23] E. P. Gross, Analytical methods in the theory of electron lattice interactions, *Ann. Phys. (N.Y.)* **8**, (1959) 78–90.

[24] E. P. Gross, Particle-like solutions in field thory, *Ann. Phys. (N.Y.)* **19**, (1962) 219–233.

[25] M. Hirokawa, Recent developments in mathematical methods for models in nonrelativistic quantum electrodinamics, *A Garden of Quanta. Essays in Honor of Hiroshi Ezawa*, eds., J. Arafune, A. Arai, M. Kobayashi, K. Nakamura, T. Nakamura, I. Ojima, N. Sakai, A. Tonomura, and K. Watanabe, 209–242, World Scientific, 2003.

[26] M. Hirokawa, Mathematical Addendum for "Infrared Catastrophe for Nelson's Model" (mp_arc 03-512), mp_arc 03-551 (2003).

[27] M. Hirokawa, Infrared catastrophe for Nelson's model. — Non-existence of ground state and soft-boson divergence —, *Publ. Ris. Inst. Math. Sci.* **42**, (2006) 897–922.

[28] M. Hirokawa, A Mathematical Mechanism of Infrared Catastrophe, mp_arc 04-83 (2004).

[29] M. Hirokawa, F. Hiroshima, and H. Spohn, Ground state for point particles interacting through a massless scalar bose field, *Adv. in Math.* **191**, (2005) 424–459.

[30] M. Hirokawa and F. Hiroshima, Note on spin-boson model through a Poisson point process, preprint (2007).

[31] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211**, (2000) 585–613.

[32] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. H. Poincaré* **3**, (2002) 171–201.

[33] F. Hiroshima, Multiplicity of ground states of quantum field models: applications of asymptotic fields, *J. Funct. Anal.* **224**, (2005) 431–470.

[34] R. Houdré, C. Weisbuch, R. P. Stanley, U. Oesterle, P. Pellandini, and M. Ilegems, Measurement of Cavity-Polariton Dispersion Curve from Angle-Resolved Photoluminescene, *Phys. Rev. Lett.* **73**, (1994) 2043–2046.

[35] H. Ibach and H. Lüth, *Solid-State Physics*, Springer-Verlag, 1990.

[36] A. Jaffe and C. Jäkel, An exchange identity for non-linear fields, *Commun. Math. Phys.* **264**, (2006) 283–289.

[37] C. Kittel, *Introduction to Solid State Physics*, John Wiley, 1990.

[38] T. D. Lee, F. E. Low, and D. Pines, The motion of slow electrons in a ploar crystal, *Phys. Rev.* **90**, (1953) 297–302.

[39] T. D. Lee and D. Pines, Interaction of a Nonrelativistic Particle with a Scalar Field with Application to Slow Electrons in Polar Crystals, *Phys. Rev.* **92**, (1953) 883–889.

[40] C. Liberti and R. L. Zaffino, Critical properties of two-level atom systems interacting with a radiation field, *Phys. Rev. A* **70**, (2004) 033808.

[41] M. H. Lieb and M. Loss, Existence of atoms and molecules in non-relativistic quantum electrodynamics, *Adv. Theo. Math. Phys.* **7**, (2003) 667–710.

[42] J. Lőrinczi, R. A. Minlos and H. Spohn, The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3**, (2002) 269–295.

[43] J. S. Møller, The polaron revisited, *Rev. Math. Phys.* **18**, (2006) 485–517.

[44] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5**, (1964) 1190–1197.

[45] A. Panati, Existence and non existence of a ground state for the massless Nelson model under binding condition, arXiv:math-ph/0609065, 2006.

[46] W. Pauli and M. Fierz, Zur Theorie der Emission langwelliger Lichtquanten, *Nuovo Cimento* **15**, (1938) 167–187.

[47] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980.

[48] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II*, Academic Press, 1980.

[49] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.

[50] R. Saito, A. Jorio, A. G.Souza Filho, G. Dresselhaus, M. S. Dresselhaus, and M. A. Pimenta, Probing phonon dispersion relations of graphite by double resonance Raman scattering, *Phys. Rev. Lett.* **88**, (2002) 027401.

[51] I. Sasaki, Ground state of the massless Nelson model in a non-Fock representation, *J. Math. Phys.* **46**, (2005) 102107.

[52] M. Sato and K. Abe, Acoustic phonon dispersion in $NiTe_2$, *J. Phys. C: Solid State Physics* **12**, (1979) L613–L615.

[53] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Birkhäuser, 1990.

[54] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (2nd Ed.), Harper and Row, 1962.

[55] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44**, (1998) 9–16.

[56] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, 1980.

[57] S. Yoshioka, Y.Tsujii, and T. Yagi, The B_2 polariton mode in KDP studied by impulsive stimulated Raman scattering, *J. Phys. Soc. Japan* **67**, (1998) 2178–2181.

Masao Hirokawa
Graduate School of Natural Science and Technology,
Okayama University,
700-8530 Okayama,
Japan
e-mail: `hirokawa@math.okayama-u.ac.jp`